The Subtree Polynomial

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Joint work with Jason Brown (Dalhousie University)

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- Alternatively,

$$\Phi_T(x) = \sum_{k=1}^{n} s_k(T) x^k,$$

where $s_k(T)$ is the number of subtrees of T of order k.

AN EXAMPLE

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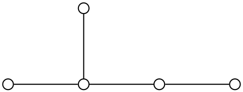

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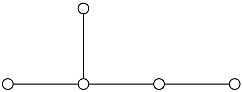

$$\Phi_T(x) = 5x$$

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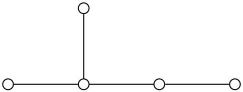

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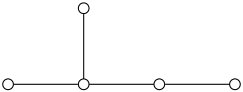

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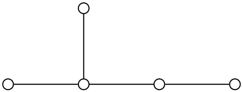

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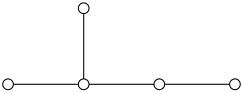

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- Less obvious: $-\Phi_T(-1)$ is the *independence number* of T (Jamsion, 1987).

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Evidently, we have:

$$\Phi_T(x) = \Phi_{T,v}(x) + \Phi_{T-v}(x).$$

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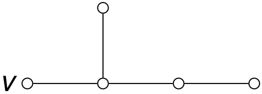

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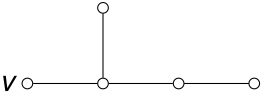

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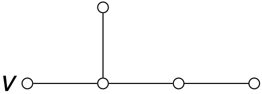

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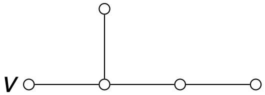

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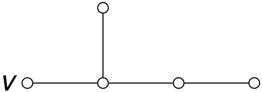

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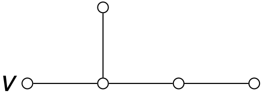

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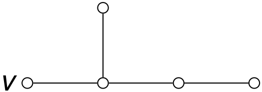

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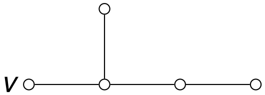

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Notice: Many of the “large” subtrees of T contain v, while many of the “small” subtrees of T do not contain v.

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INTRODUCTION

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COMPLEX SUBTREE ROOTS

REAL SUBTREE ROOTS

OPEN PROBLEMS

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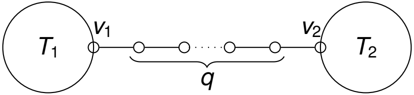
- We give a short proof of this result.

CSIKVÁRI'S GENERALIZED TREE SHIFT

Let T be a tree, and let v_1 and v_2 be non-leaf vertices of T connected by a path whose internal vertices all have degree 2.



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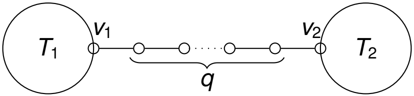
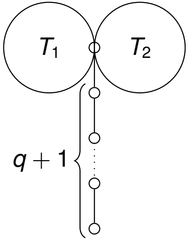


The following tree T' is said to be obtained from T by a *generalized tree shift*:



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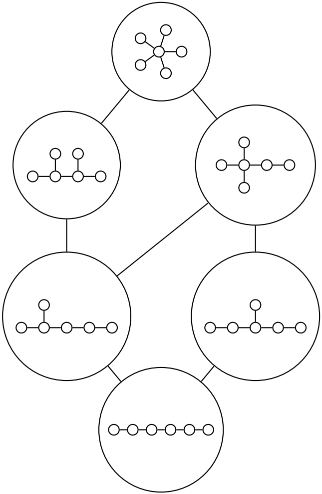

Theorem: Let T be a tree of order $n \geq 4$, and let T' be a tree obtained from T by a generalized tree shift. For every $k \in \{3, \ldots, n-1\}$, $s_k(T) < s_k(T')$.

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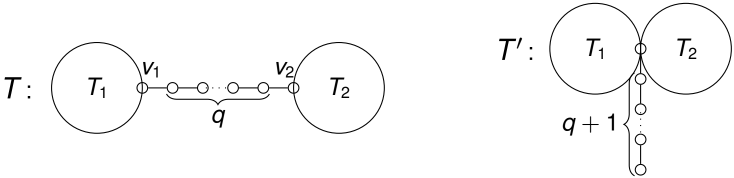

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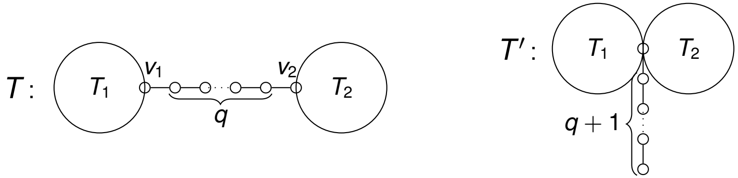

Sketch of Proof: Show that

$$\Phi_{T'}(x) - \Phi_T(x) = \frac{1}{x} \cdot \left[\sum_{i=0}^{q} x^i\right] \cdot \left[\Phi_{T_1,v_1}(x) - x\right] \cdot \left[\Phi_{T_2,v_2}(x) - x\right].$$

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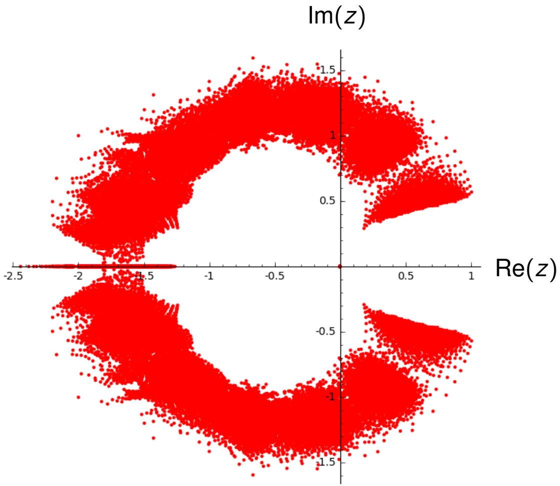

The subtree roots of all trees of order at most 14.

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- This bound is tight! The only tree with a subtree root of modulus $1 + \sqrt[3]{3}$ is the star $K_{1,3}$.

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This is proven using a rather technical strong induction. The key is a good lower bound on $|\Phi_{T,v}(z)|$.

Lemma: Let T be a tree of order n with vertex v. If $|z| \geq 2$, then

$$|\Phi_{T,v}(z)| \geq |z| \cdot (|z| - 1)^{n-1} .$$

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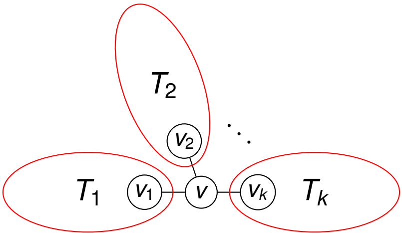

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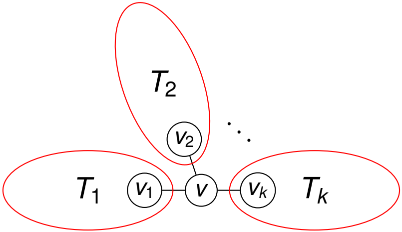

$$\Phi_{T,v}(z) = z \cdot \prod_{i=1}^{k} \left[1 + \Phi_{T_i,v_i}(z)\right]$$

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- The subtree polynomial has no complex roots of modulus greater than $1 + \sqrt[3]{3}$
  - $\Rightarrow$ no subtree roots in $\left(-\infty, -1 - \sqrt[3]{3}\right)$
- Question: What about the interval $(-1 - \sqrt[3]{3}, 0)$?

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- **Useful identity:** $x\Phi'_T(x) = \sum_{v \in V(T)} \Phi_{T,v}(x)$
- **Claim:** $\forall v \in V(T)$, $x \in (-1, 0) \Rightarrow \Phi_{T,v}(x) \in (-1, 0)$.
- Claim is proved using induction and the nice recursive formula for $\Phi_{T,v}(x)$:

$$\Phi_{T,v}(x) = x \cdot \prod_{i=1}^{k} \left[1 + \Phi_{T_i,v_i}(x)\right]$$

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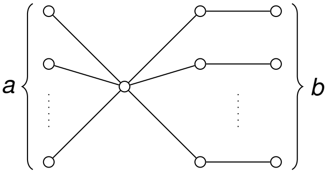
Proof technique: We show that the collection of real subtree roots of the following family of trees is dense in $[-2, -1]$:



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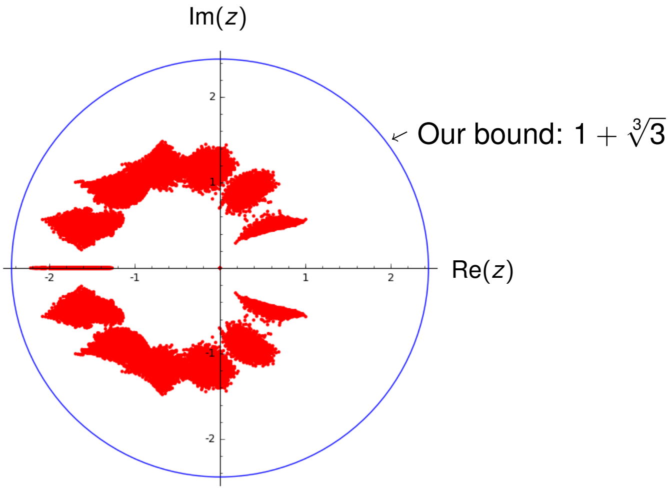
Find a tighter bound on the subtree roots of all trees of order *n*.



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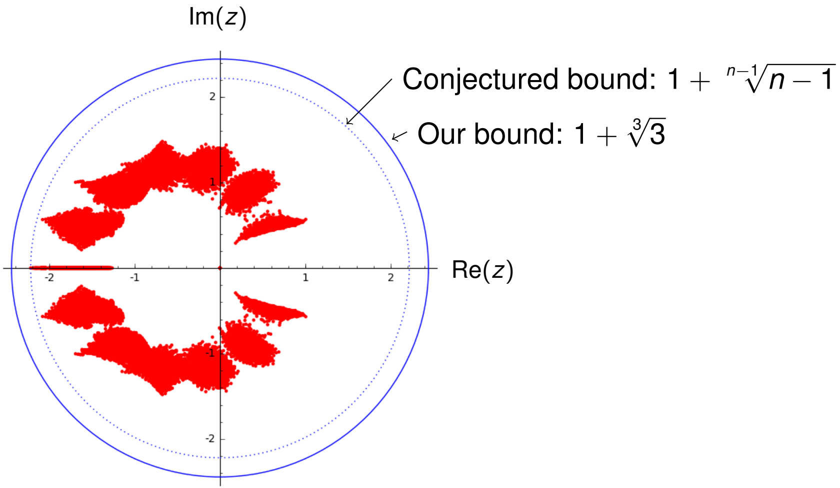
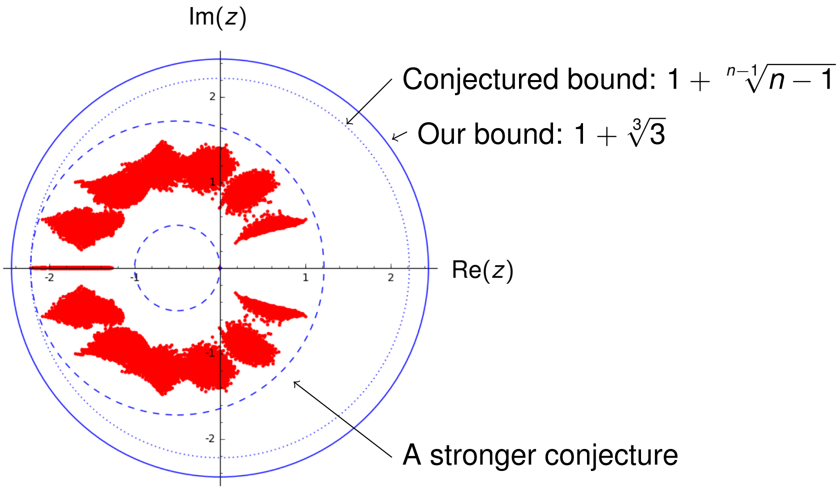


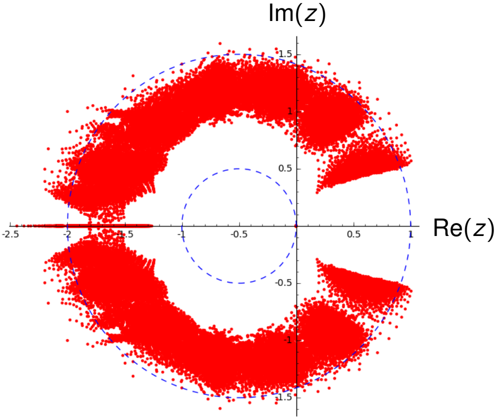
Open problems

Find a tighter bound on the subtree roots of all trees of order n.



CLOSURE

Does the closure of the collection of subtree roots contain the entire annulus $\frac{1}{2} \leq \left|z + \frac{1}{2}\right| \leq \frac{3}{2}$?



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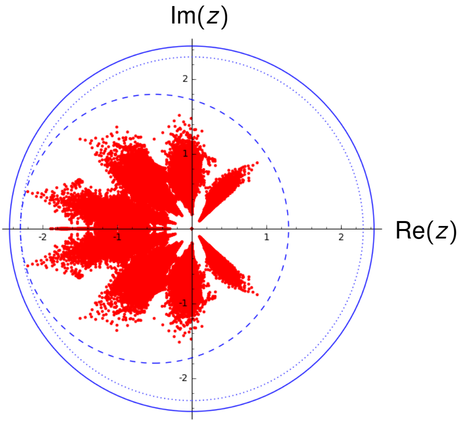

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- For which trees is the sequence of coefficients $s_2(T), s_3(T), \ldots, s_n(T)$ unimodal?
- In particular, is this sequence unimodal when T has no vertices of degree 2?
  - Fact: This sequence is NOT log-concave for all such trees.

Thank you!