On the roots of Wiener polynomials of graphs

Lucas Mol

THE UNIVERSITY OF WINNIPEG

Joint work with Jason Brown (Dalhousie University) and
Ortrud Oellermann (University of Winnipeg)

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Plan

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- When context is clear, we use $d(u,v)$ and D instead.

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- The related concept of *average distance* has also proven useful in many applications, including architecture.

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- Equivalently,

$$W(G; x) = \sum x^{d(u,v)},$$

where the sum is over all pairs of vertices u, v of G.

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- The Wiener polynomial is an analytic tool for exploring the sequence of coefficients $d_1, d_2, \ldots, d_D$.

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- Real Wiener roots: What is the closure of the collection of real Wiener roots of all graphs?
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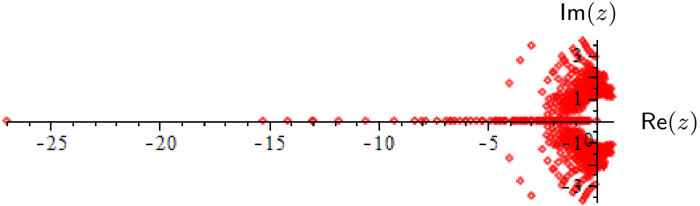
We also consider each of these questions for the collection of trees.

Plan

Wiener roots of all graphs of order 8



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$$r \leq |z| \leq R$$

where $r = \min\left\{\frac{a_i}{a_{i+1}}\right\}$ and $R = \max\left\{\frac{a_i}{a_{i+1}}\right\}$.

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- To apply this theorem, we need to bound the ratios of successive coefficients of the Wiener polynomial.
- This is an interesting problem in its own right, as it tells us about the sequence of coefficients $d_1, d_2, \ldots, d_D$.

Theorem (BMO, 2018)

The maximum modulus among Wiener roots of connected graphs of order $n \geq 2$ is $\binom{n}{2} - 1$. Moreover, for all $n \geq 3$, the graph $K_n - e$ (where e is any edge of K_n) is the unique graph of order n with a Wiener root of this modulus.

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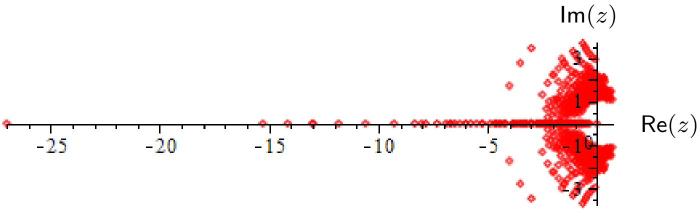

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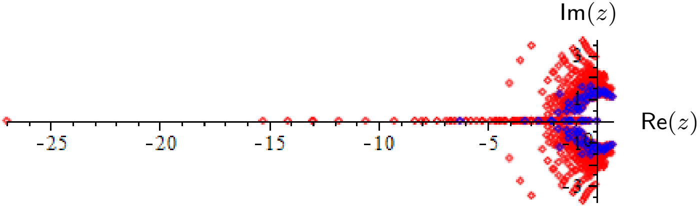

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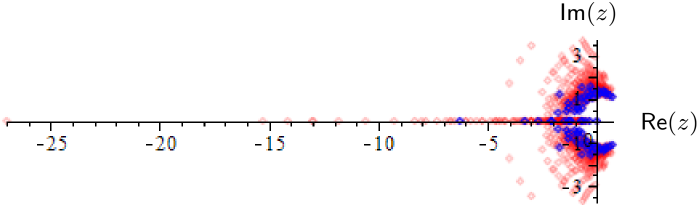

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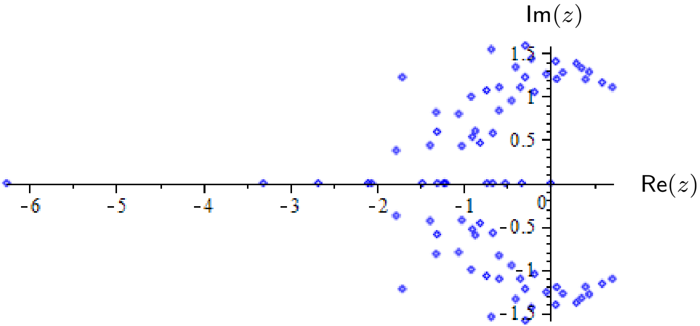

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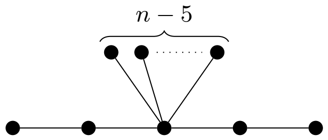

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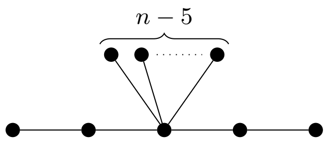

Figure: The tree T_n.

- The tree T_n has a Wiener root of modulus approximately $(1+\frac{1}{\sqrt{2}})n$.

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Among all graphs of order $n \geq 3$, the graph $K_{1,n-1}$ has nonzero Wiener root of mimimum modulus $\frac{2}{n-2}$. Moreover, $K_{1,n-1}$ is the unique graph of order n with a Wiener root of this modulus for all $n \geq 3$.

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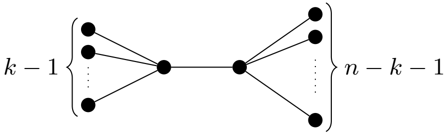

Figure: The double star $S_{k,n-k}$.

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- We show that the collection of complex Wiener roots does not lie in any half-plane (the set of all points on one side of a line).

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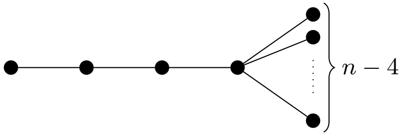

Figure: The broom $B_{4,n-4}$.

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There are connected graphs (even trees) with Wiener roots having arbitrarily large imaginary part.

Sketch of Proof.

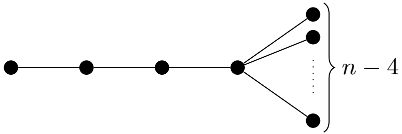

Figure: The broom $B_{4,n-4}$.

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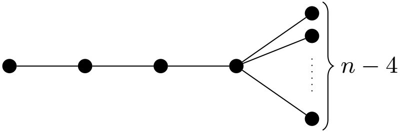

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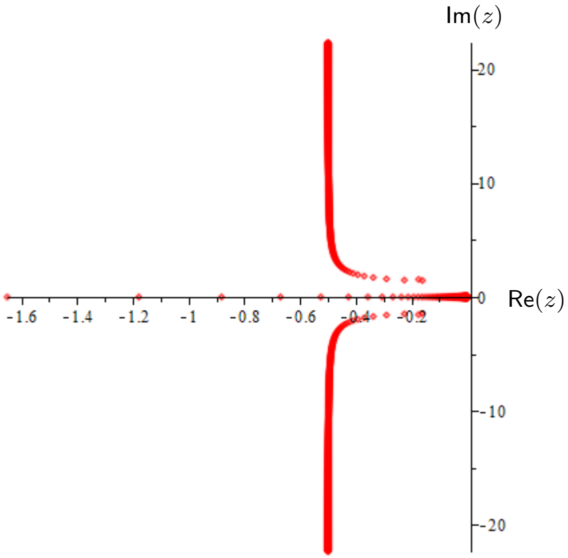
Im(z)
Re(z)
20
10
0
-10
-20
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2

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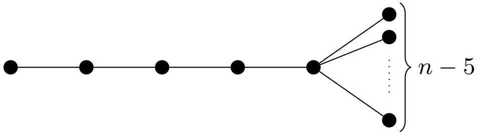

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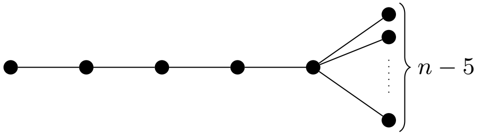

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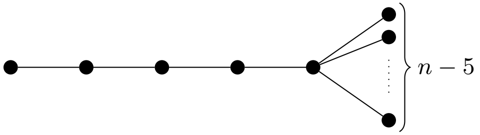

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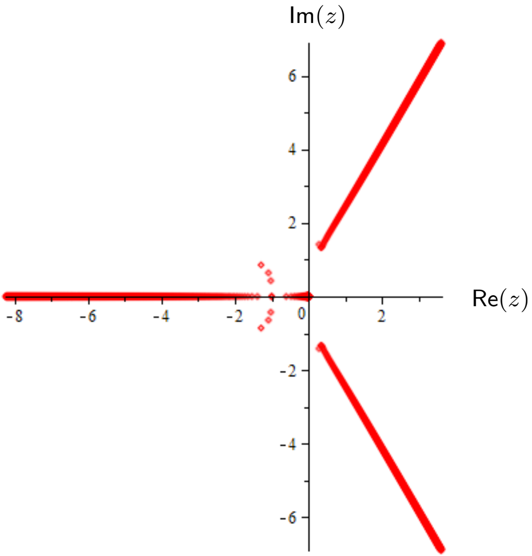
Im(z)
Re(z)
6
4
2
-2
-4
-6
-8
-6
-4
-2
0
2

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