Roots of all-terminal reliability and node reliability polynomials

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Plan

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- Simplifying assumption: All components perform with the same fixed probability $p \in (0, 1)$.

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Theorem (Brown, Mol 2017)

Let G be a 2-connected (multi)graph of order n. If $R_A(G; p) = 0$, then $|1 - p| \leq n - 1$.

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- By the Eneström-Kakeya Theorem, $|1-p| \leq \frac{H_{m-n}}{H_{m-n+1}}$.

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Corollary (Brown, Mol 2017)

Let G be a connected (multi)graph of order $n \geq 2$ in which the maximum order of a block is b. If $R_A(G;p) = 0$, then $|1-p| \leq b-1$.

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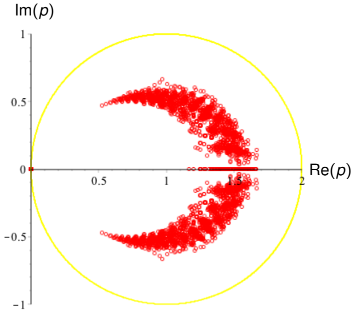

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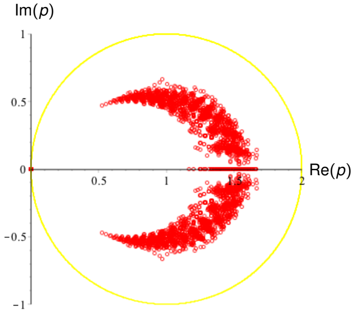

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- The *real* all-terminal reliability roots are indeed contained in $\{0\} \cup (1, 2]$ (Brown and Colbourn, 1992).

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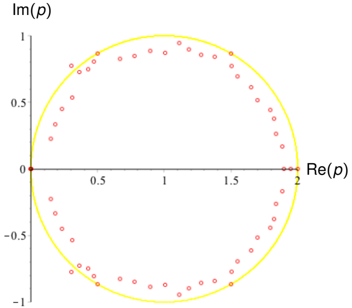

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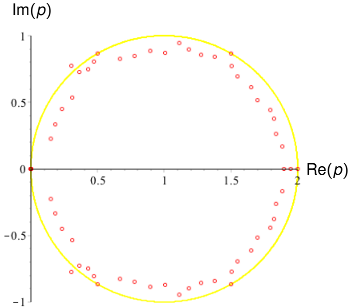

- However, the roots are not far outside of the unit disk!

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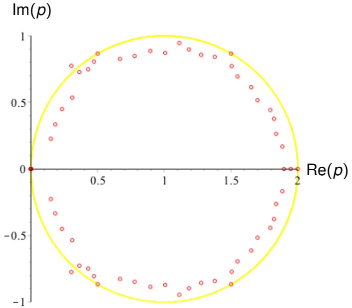
- We generalized the multigraph of Royle and Sokal, finding roots slightly further outside of the unit disk.



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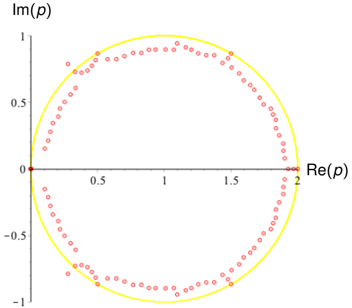

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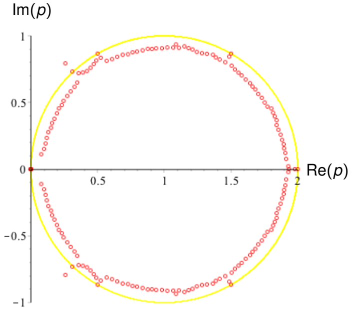

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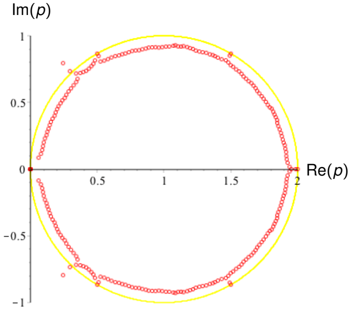

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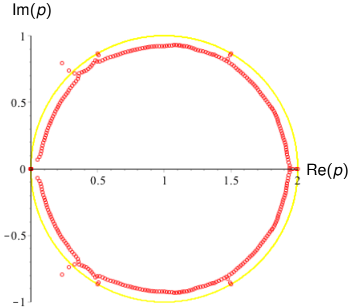

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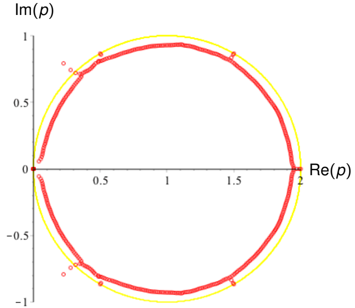

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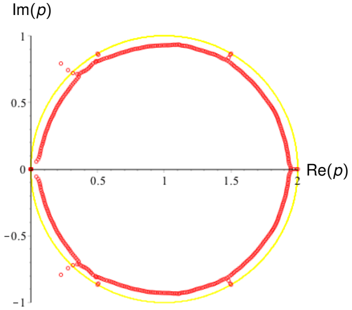

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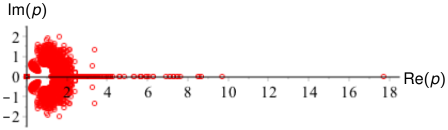

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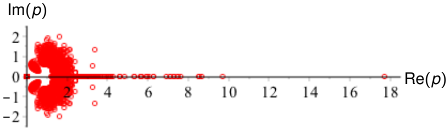

Node reliability roots of all simple graphs of order 7.

Theorem (Brown, Mol 2016)

The collection of node reliability roots is unbounded, even if we restrict to real roots. In particular, for all $n \geq 2$, $R_N(C_{2n+1}, p)$ *has a real root in the interval* $(2n^2 - 1, 2n^2)$.

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Theorem (Brown, Colbourn 1994)

Every connected graph has a subdivision whose all-terminal reliability roots are all real.

Theorem (Brown, Mol 2016)

Every connected graph on at least 3 vertices has a nonreal node reliability root.

Proof: Let G be a connected graph on $n \geq 3$ vertices, and suppose towards a contradiction that $R_N(G;p)$ has all real roots.

- Let $R_N(G;p) = \sum_{k=1}^{n} N_k p^k (1-p)^k$.
- Consider the related polynomial $C(G;x) = \sum_{k=1}^{n} N_k x^k$, which has a nonreal root iff $R_N(G;p)$ does.
- Useful result: If all zeros of $f(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}$ are real and negative, then $\frac{a_{n-1}}{a_n} \cdot \frac{a_1}{a_0} \geq (n-1)^2$.

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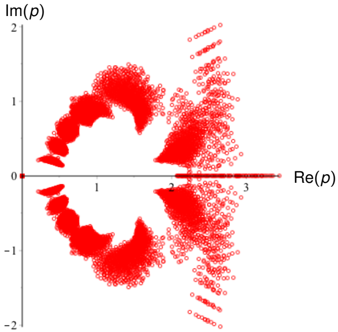

Node reliability roots of all trees of order 13.