The repetition threshold for binary rich words

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Joint work with James D. Currie and Narad Rampersad

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Plan

Critical Exponents and Repetition Thresholds

Rich words

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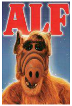

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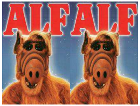

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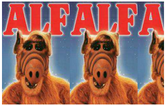

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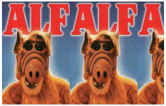
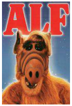


$=$ (

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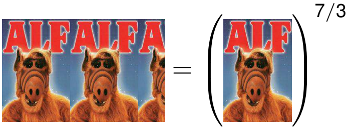

- Special case: 2-powers are also called *squares*.

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- The *repetition threshold* for a set of words L is the smallest critical exponent among all infinite words in L.
  - Since every long enough binary word contains a square, the repetition threshold for the set of all binary words is 2.

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- In particular, if w is an infinite binary word with critical exponent less than $7/3$, then it contains *every factor* of the Thue-Morse word.

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- An infinite word is called *rich* if all of its finite factors are rich.

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  - We will determine $\mathrm{RRT}(2)$.

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- The irrationality of $2 + \sqrt{2}/2$ makes this hard to prove!
- Baranwal and Shallit: $\text{RRT}(2) \geq 2.7$

Baranwal and Shallit's construction

Define morphisms f and h by

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- The proof was completed using the automatic theorem proving software `Walnut`.

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- Unfortunately, this is not the case!
- Fortunately, it is not much worse than this.

Another structure theorem

Every infinite binary rich word with critical exponent less than $14/5$ looks like either $u = f(h^\omega(\texttt{0}))$ or $v = f(g(h^\omega(\texttt{0})))$.

$$f(\texttt{0}) = \texttt{0} \qquad g(\texttt{0}) = \texttt{011} \qquad h(\texttt{0}) = \texttt{01}$$
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Theorem (Currie, Mol, and Rampersad, 2020+): Let w be an infinite rich word over the binary alphabet $\{\mathtt{0}, \mathtt{1}\}$ with critical exponent less than $14/5$. For every $n \geq 1$, a suffix of w has the form $f(h^n(w_n))$ or $f(g(h^n(w_n)))$ for some infinite word w_n over $\{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$.

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- In fact, we show that v is rich, and has critical exponent exactly $2 + \sqrt{2}/2$.

An irrational repetition threshold!

Theorem (Currie, Mol, and Rampersad, 2019+): The repetition threshold for binary rich words is $2 + \sqrt{2}/2$.

Proof:

- If an infinite binary rich word has critical exponent less than $14/5$, then it contains all factors of $u = f(h^\omega(\texttt{0}))$ or all factors of $v = f(g(h^\omega(\texttt{0})))$.
- Baranwal and Shallit showed that the critical exponent of u is $2 + \sqrt{2}/2$.
- So it suffices to show that v has critical exponent at least $2 + \sqrt{2}/2$.
- In fact, we show that v is rich, and has critical exponent exactly $2 + \sqrt{2}/2$.
- Our proof technique can also be applied to u, providing an alternate proof of Baranwal and Shallit's result.

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- By a theorem of Rote (1994), this means that u and v are *complementary symmetric Rote words*.
- By a theorem of Blondin-Massé et al. (2011), every complementary symmetric Rote word is rich.
- Therefore, both u and v are rich!

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- Remember that $\Delta(v)$ is a Sturmian word.
- We can apply general results on repetitions in Sturmian words to establish the critical exponent of v.

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Summary

- Every infinite binary rich word with critical exponent less than $14/5$ looks like either *u* or *v*.
- Both *u* and *v* are complementary symmetric Rote words; we use this fact to prove that they are rich and have critical exponent $2 + \sqrt{2}/2$.
- We conclude that the repetition threshold for binary rich words is $2 + \sqrt{2}/2$.

FUTURE PROSPECTS

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- Determining the repetition threshold for rich words on $k > 2$ letters remains an open problem.
  - Is $\mathrm{RRT}(k)$ rational for $k > 2$?
  - Is $\lim_{k \to \infty} \mathrm{RRT}(k) = 2$?

More about the Structure Theorem

$$f(0) = 0 \qquad g(0) = 011 \qquad h(0) = 01$$
$$f(1) = 01 \qquad g(1) = 0121 \qquad h(1) = 02$$
$$f(2) = 011 \qquad g(2) = 012121 \qquad h(2) = 022$$

Theorem (Currie, Mol, and Rampersad, 2020+): Let w be an infinite rich word over the binary alphabet $\{0, 1\}$ with critical exponent less than $14/5$. For every $n \geq 1$, a suffix of w has the form $f(h^n(w_n))$ or $f(g(h^n(w_n)))$ for some infinite word w_n over $\{0, 1, 2\}$.

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Idea of Proof: Suppose w is an infinite binary rich word with critical exponent less than $14/5$, e.g.,

$$w = 1001100100110110010011 \cdots$$

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So some suffix of w can be written in the form $f(w')$.

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- Obviously, the word w' must be cube-free.
- So this gives us a large set of forbidden factors in w'.
- Divide into two cases:
  - w' contains the factor `0110`.
  - w' does not contain the factor 0110.

$f(\texttt{0}) = \texttt{0}$ $g(\texttt{0}) = \texttt{011}$ $h(\texttt{0}) = \texttt{01}$

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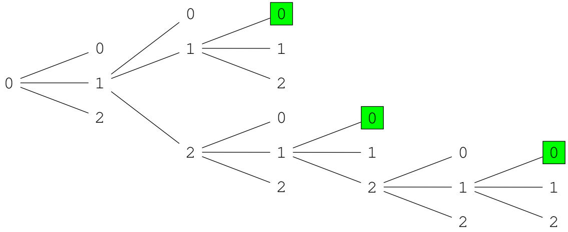

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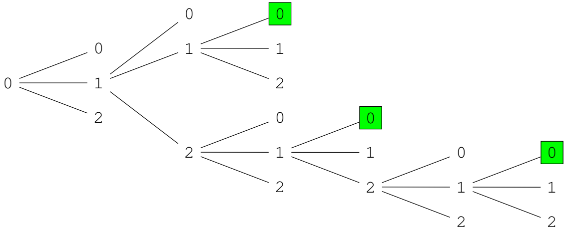

- Show that the word ending at every unboxed leaf of this tree contains a forbidden factor.

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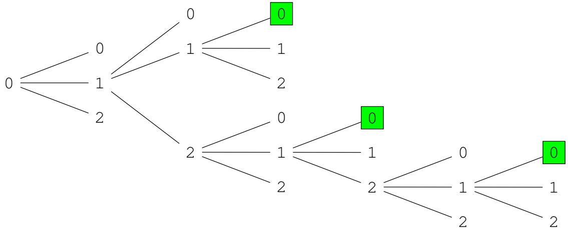

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- So a suffix of w' can be written in the form $f(g(w''))$.

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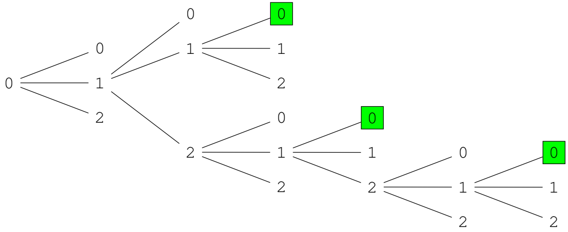

- Show that the word ending at every unboxed leaf of this tree contains a forbidden factor.
- So a suffix of w' can be written in the form $f(g(w''))$.
- Apply a similar argument to show that some suffix of w'' can be written in the form $f(g(h(w_1)))$.

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- So altogether, we see that w has a suffix of the form $f(g(h(w_1)))$, or a suffix of the form $f(h(w_1))$.
- This completes the base case of an inductive proof.
- The inductive step is proved by a similar (though slightly more technical) unified argument.

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where

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- This suggests that $14/5$ is indeed the largest possible constant for which the structure theorem holds.

Thank you!