The repetition threshold for binary rich words

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THE UNIVERSITY OF WINNIPEG

Joint work with James D. Currie and Narad Rampersad

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PLAN

WORDS AND REPETITIONS

RICH WORDS

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    `0`, `1`, `01`, `11`, `10`, `011`, `110`, and `0110`, but NOT `00`.

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  - There is no such word over Σ_2.

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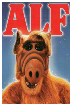

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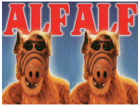

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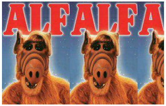

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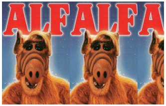
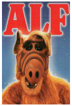


$= \Big($ 

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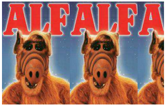

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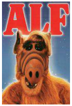

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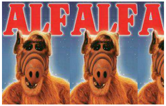

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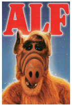

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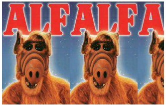
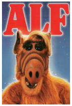


$= \Big($  $\Big)^{7/3}$

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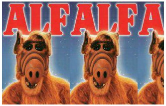
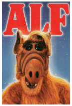


$=$ (

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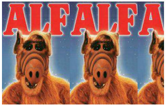

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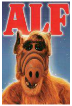

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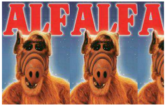

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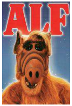

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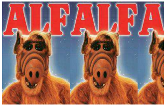
$$\mu^\omega(0) = \texttt{011}\textcolor{red}{\texttt{1010}}\textcolor{green}{\texttt{0}}\texttt{110010110}\cdots$$

- This word contains no cubes.
- In fact, it contains no *fractional powers* larger than 2.
  - Example: `alfalfa` is a $7/3$-power



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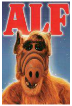

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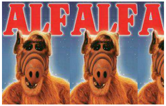

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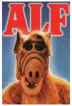

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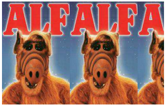
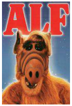


$=$ () $^{7/3}$

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Theorem (Karhumäki and Shallit, 2004): Let w be an infinite binary word with critical exponent less than $7/3$. For every $n \geq 1$, a suffix of w has the form $\mu^n(w_n)$ for some infinite binary word w_n.

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Plan

Words and Repetitions

Rich words

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- An infinite word is called *rich* if all of its finite factors are rich.

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- Baranwal and Shallit: $\text{RRT}(2) \geq 2.7$

BARANWAL AND SHALLIT'S CONSTRUCTION

Define morphisms f and h by

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f(\texttt{0}) &= \texttt{0} \\
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- The proof was completed using the automatic theorem proving software `Walnut`.

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- Fortunately, it is not much worse than this.

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Every infinite binary rich word with critical exponent less than $14/5$ looks like either $u = f(h^\omega(\mathtt{0}))$ or $v = f(g(h^\omega(\mathtt{0})))$.

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Theorem (Currie, Mol, and Rampersad, 2019+): Let w be an infinite rich word over the binary alphabet $\{\mathtt{0}, \mathtt{1}\}$ with critical exponent less than $14/5$. For every $n \geq 1$, a suffix of w has the form $f(h^n(w_n))$ or $f(g(h^n(w_n)))$ for some infinite word w_n over $\{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$.

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- We handle v.
- Our proof technique can also be applied to u, providing an alternate proof of Baranwal and Shallit's result.

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- By a theorem of Blondin-Massé et al., every complementary symmetric Rote word is rich.
- Therefore, both u and v are rich!

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SUMMARY

- Every infinite binary rich word with critical exponent less than $14/5$ looks like either u or v.
- Both u and v are complementary symmetric Rote words; we use this fact to prove that they are rich and have critical exponent $2 + \sqrt{2}/2$.
- We conclude that the repetition threshold for binary rich words is $2 + \sqrt{2}/2$.

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  - Is $\lim_{k \to \infty} \mathrm{RRT}(k) = 2$?

Thank you!