The Mean Subtree Order and the Mean Connected Induced Subgraph Order

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THE UNIVERSITY OF WINNIPEG

Joint work with Kristaps Balodis and Ortrud Oellermann (The University of Winnipeg), and Matthew Kroeker (The University of Waterloo)

CanaDAM 2019 – Average Graph Parameters Minisymposium

PLAN

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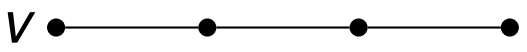

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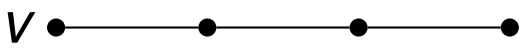

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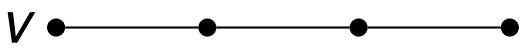

$$M_T = \frac{20}{10} = 2 \qquad M_{T,v} = \frac{10}{4} = \frac{5}{2}$$

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- For example, $\lim_{n\to\infty} \operatorname{den}(P_n) = \frac{1}{3}$.

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  - It follows that the proportion of vertices of T_k of degree 2 must approach 1.

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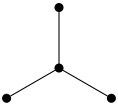

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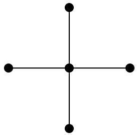

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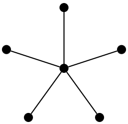

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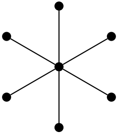

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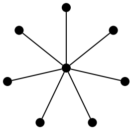

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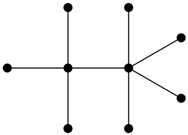

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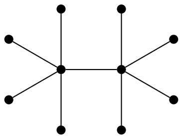

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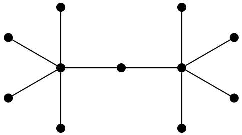

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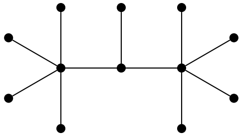

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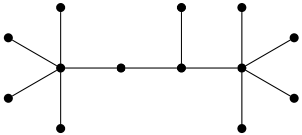

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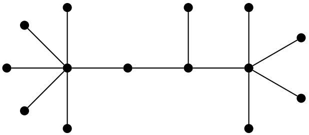

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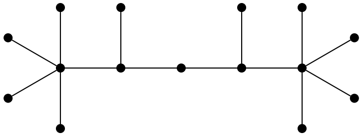

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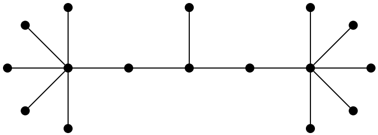

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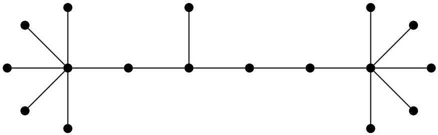

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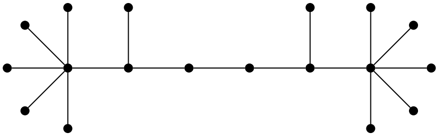

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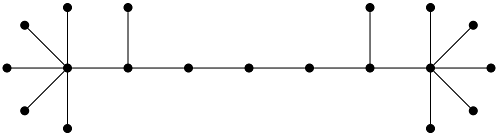

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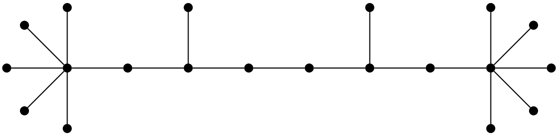

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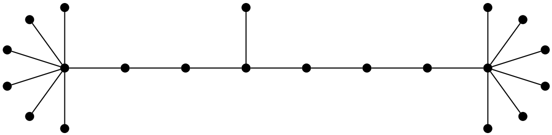

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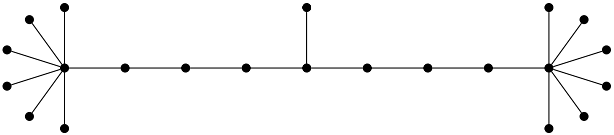

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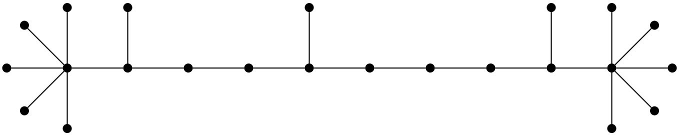

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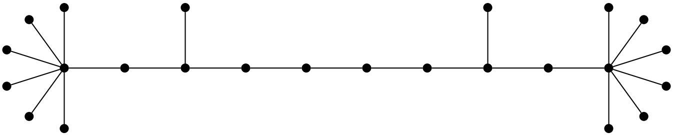

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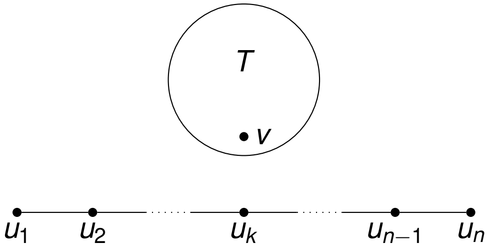
Let T be a tree of order at least 2 with vertex v.



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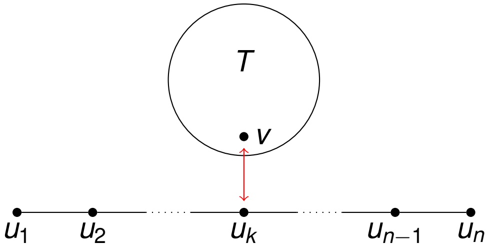

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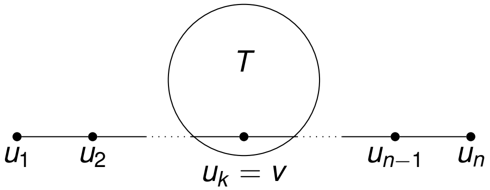

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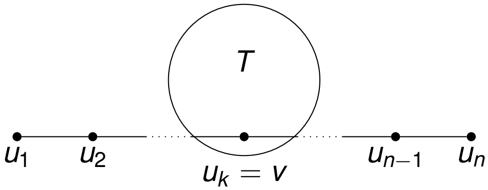

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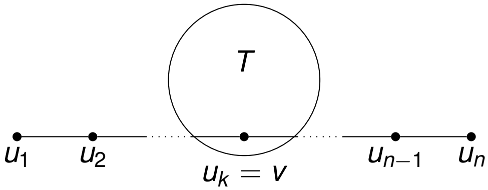

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The Gluing Lemma (Mol and Oellermann, 2018):
If $1 \leq r < s \leq \frac{n+1}{2}$, then $M_{T_r} < M_{T_s}$.

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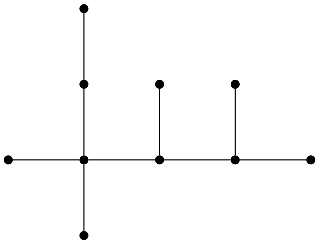

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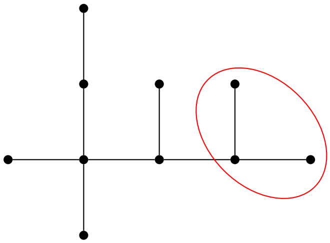

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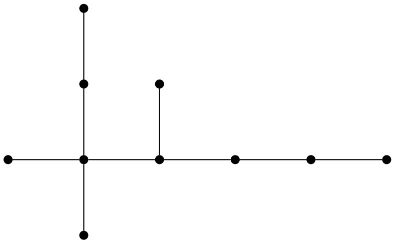

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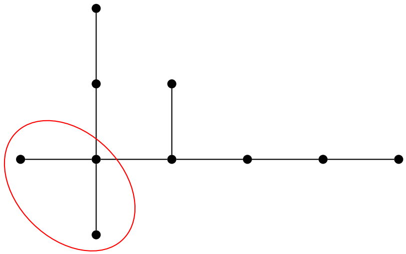

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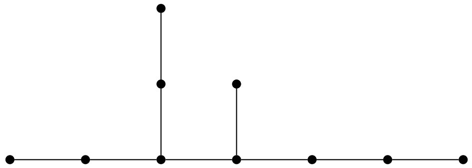

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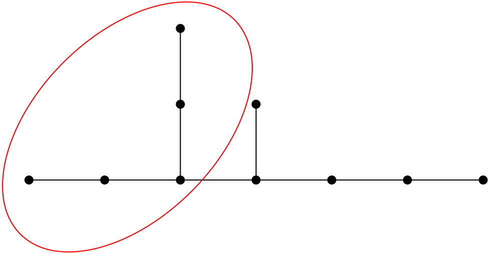

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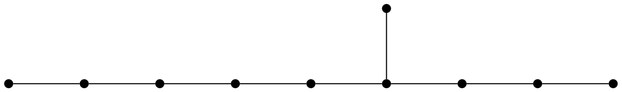

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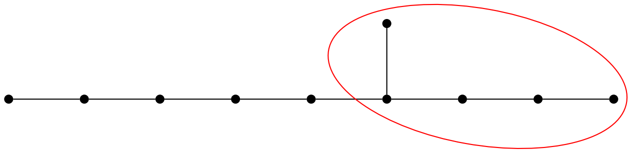

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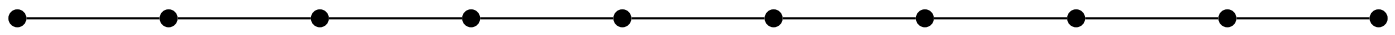

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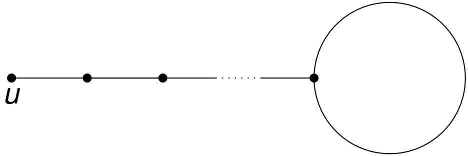

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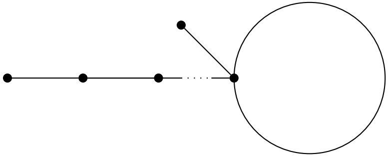

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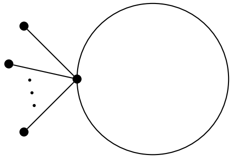

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From now on, M_G and $M_{G,v}$ denote the global and local versions, respectively, of the mean connected induced subgraph order (mean CIS order).

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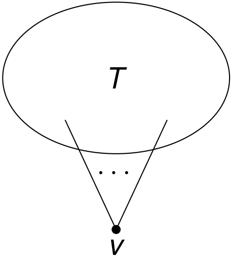

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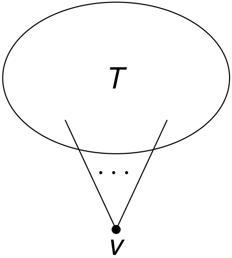

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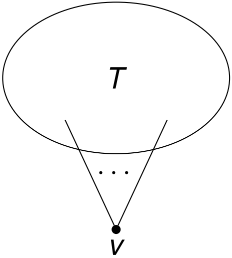

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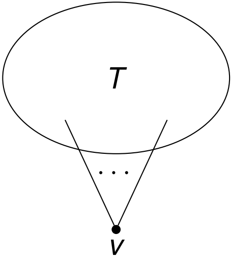

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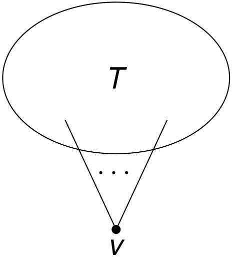

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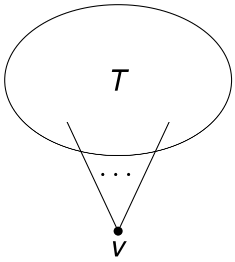

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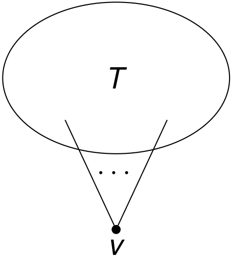

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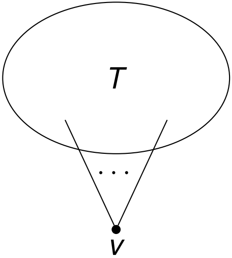

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Challenge: Many important results for trees rely on the local/global mean inequality!

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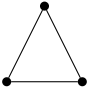

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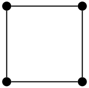

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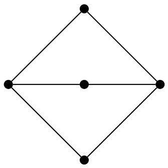

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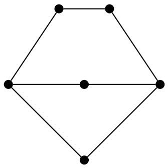

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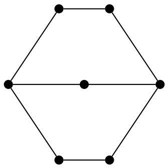

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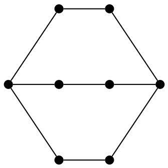

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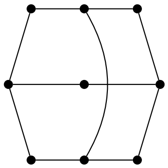

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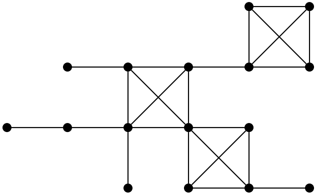

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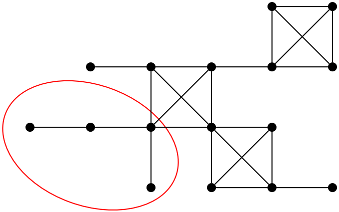

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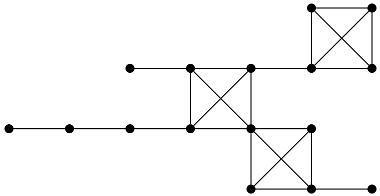

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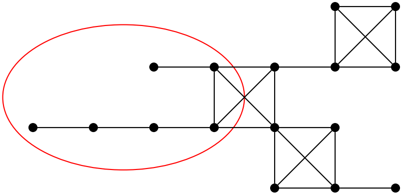

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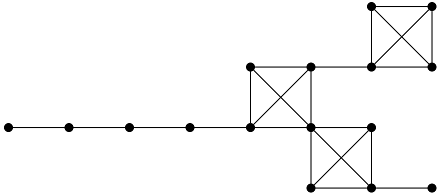

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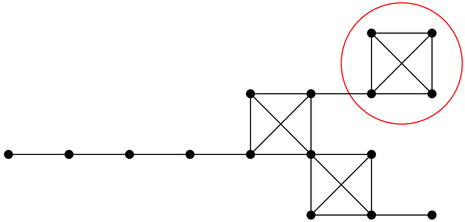

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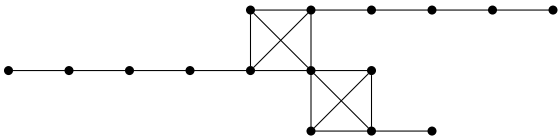

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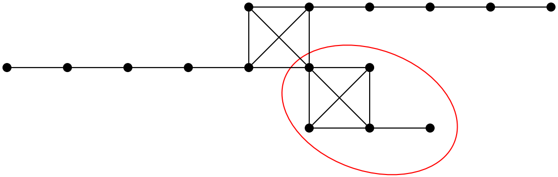

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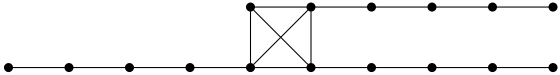

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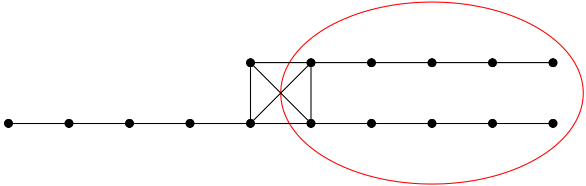

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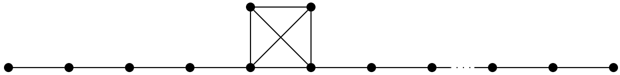

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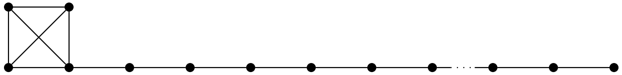

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- We think that the path has minimum mean CIS order among all connected graphs of order n. How can we prove this?

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- We know that there are trees with density arbitrarily close to 1. What more can we say about trees of order n with maximum mean subtree order?
  - Consider sticking around for the next talk.
- We think that the path has minimum mean CIS order among all connected graphs of order n. How can we prove this?
- What can we say about the connected graphs of order n with largest mean CIS order?

Thank You!