The Mean Subtree Order of Graphs Under Edge Addition

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Joint work with Ben Cameron (University of Guelph)

Average Graph Parameters Minisymposium
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PLAN

MEAN SUBTREE ORDER OF TREES

MEAN SUBTREE ORDER OF GRAPHS

CONCLUSION

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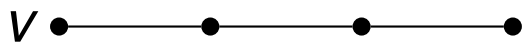

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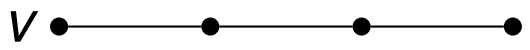

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- Conjecture: The tree of order n with maximum mean subtree order is a caterpillar.

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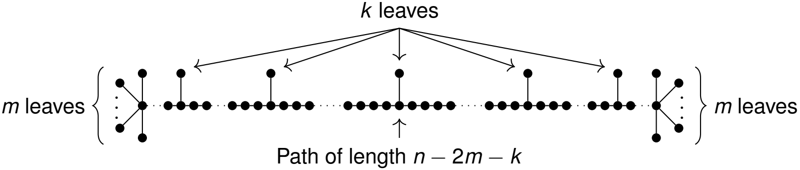

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We will focus on the first extension for the remainder of this talk.

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- Thus we have

$$M_G = \frac{\Phi'_G(1)}{\Phi_G(1)},$$

the logarithmic derivative of $\Phi_G(x)$ evaluated at 1.

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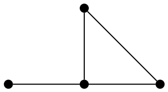

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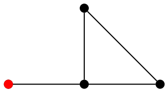

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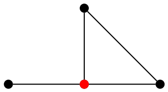

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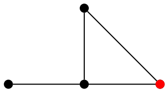

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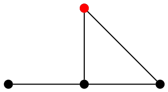

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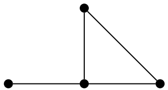

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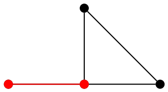

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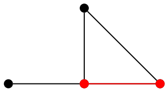

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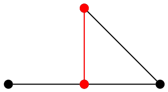

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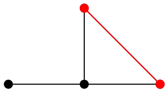

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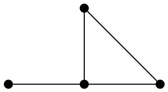

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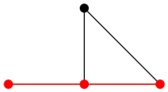

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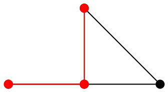

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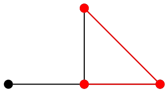

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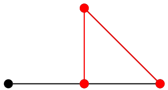

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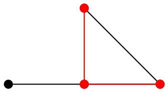

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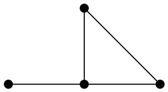

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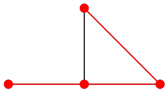

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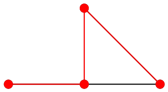

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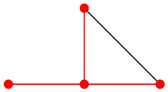

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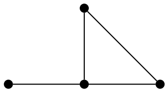

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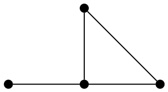

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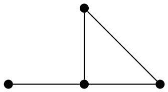

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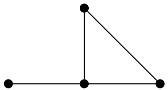

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- Local mean subtree order at a vertex (or an edge) can be computed analogously with a local subtree polynomial.

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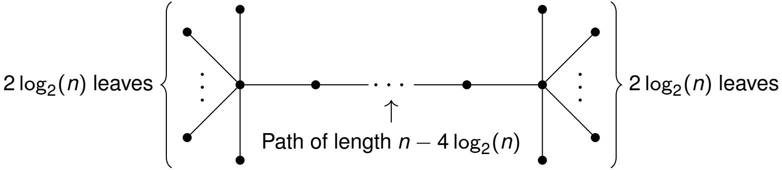

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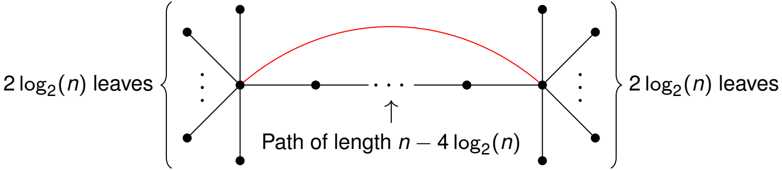

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- We have a proof in the special case that G is a tree.

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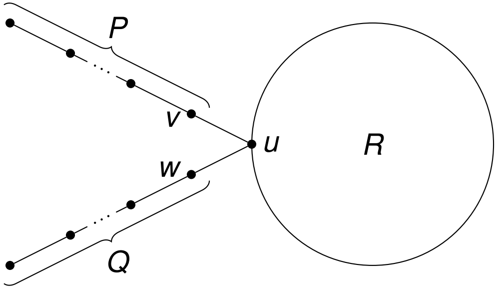
Idea of Proof: Find a vertex u in T that looks like this:



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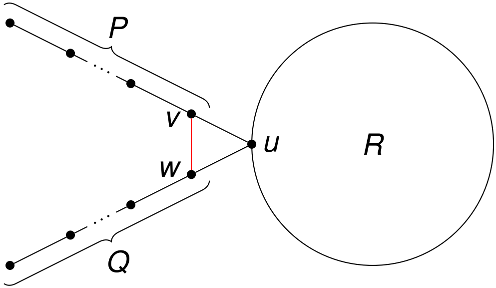
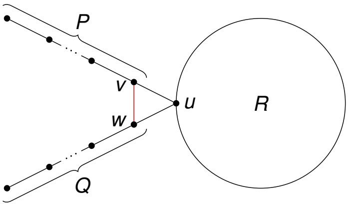
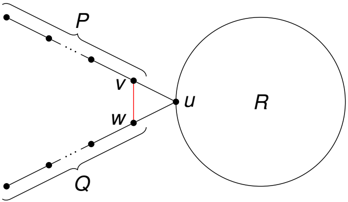


Let H be the graph obtained from T by adding the edge e between v and w.



$$\Phi_H(x) = \Phi_{H,e}(x) + \Phi_T(x)$$

- $\Phi_H(x)$: Subtrees of H
- $\Phi_{H,e}(x)$: Subtrees of H that contain e
- $\Phi_T(x)$: Subtrees of H that don't contain e



$$\Phi_H(x) = \Phi_{H,e}(x) + \Phi_T(x)$$

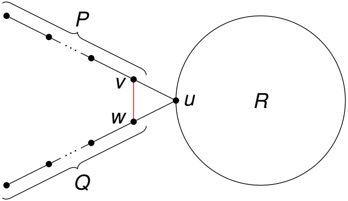
Subtrees of H

Subtrees of H that contain e

Subtrees of H that don't contain e

- So M_H is a weighted average of $M_{H,e}$ and M_T:

$$M_H = \frac{\Phi_{H,e}(1)}{\Phi_H(1)} M_{H,e} + \frac{\Phi_T(1)}{\Phi_H(1)} M_T.$$



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Subtrees of H	Subtrees of H that contain e	Subtrees of H that don't contain e

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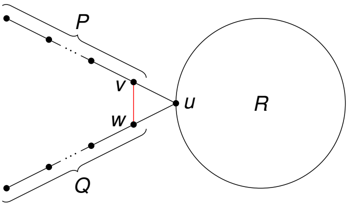
- So it suffices to show that $M_{H,e} > M_T$.



$$\Phi_H(x) = \Phi_{H,e}(x) + \Phi_T(x)$$

Subtrees of H | Subtrees of H that contain e | Subtrees of H that don't contain e

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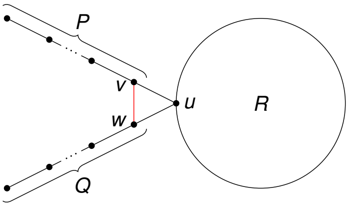
- Jamison's local/global mean inequality: $M_{T,u} > M_T$.



$$\Phi_H(x) = \Phi_{H,e}(x) + \Phi_T(x)$$

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- Jamison's local/global mean inequality: $M_{T,u} > M_T$.
- So it suffices to show that $M_{H,e} \geq M_{T,u}$.

Plan

Mean Subtree Order of Trees

Mean Subtree Order of Graphs

Conclusion

Open Problems

- Conjecture: Among all connected graphs of order n, the graph K_n has maximum mean subtree order, and the path P_n has minimum mean subtree order.

One way to prove this conjecture would be to prove the following two conjectures.

- Conjecture: Let G be a connected graph not isomorphic to a complete graph. Then there is some non-edge e of G such that $M_{G+e} > M_G$.
- Conjecture: Let G be a connected graph that is not a tree. Then there is an edge e of G such that e is not a bridge and $M_{G-e} < M_G$.

Thank You!