Circular repetition thresholds for small alphabets: Last cases of Gorbunova's Conjecture

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The University of Winnipeg

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Plan

Background

Four letters

Five letters

Conclusion

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- The length of word w is denoted $|w|$.

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- We will mostly be interested in factors of exponent β where $1 < \beta < 2$, which can always be written as xyx with $|xyx|/|xy| = \beta$.

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Further, every sufficiently long word on two letters contains a factor of exponent 2 (a *square*).

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Let $k \geq 2$. The *repetition threshold* for k letters, denoted $\mathrm{RT}(k)$, is the infimum of the set of all β such that there are β-free words of every length on k letters.

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Conjecture (Dejean, 1972)

$$\mathrm{RT}(k) = \begin{cases} \frac{7}{4} & \text{if } k = 3 \\ \frac{7}{5} & \text{if } k = 4 \\ \frac{k}{k-1} & \text{if } k \geq 5. \end{cases}$$

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- As a linear word, `onion` is 2-free.
- However, the circular word (`onion`) has factor `onon`, so it is not 2-free.

Circular Repetition Threshold

Definition

Let $k \geq 2$. The *circular repetition threshold* for k letters, denoted $\mathrm{CRT}(k)$, is the infimum of the set of all β such that there are β-free circular words of every length on k letters.

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- Last remaining cases: $\mathrm{CRT}(4)$ and $\mathrm{CRT}(5)$.

Last Cases of Gorbunova's Conjecture

k	$\mathrm{RT}(k)$	$\mathrm{CRT}(k)$
2	2	5/2
3	7/4	2
4	7/5	3/2
5	5/4	4/3
6	6/5	4/3
7	7/6	5/4
8	8/7	5/4
9	9/8	6/5
10	10/9	6/5

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- $\frac{4}{3}^+$-free circular words of every length on 5 letters.

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- Iterating gives β-free words of arbitrarily long length.

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Solution:

- Use two different morphisms: an r-uniform morphism and an s-uniform morphism (where r and s are relatively prime).

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- Take a $\frac{3}{2}^+$-free circular word (w) of length $k + \ell$, and write it as $w = uv$, where $|u| = k$ and $|v| = \ell$.
- Claim: $(f_9(u)f_{11}(v))$ is $\frac{3}{2}^+$-free.

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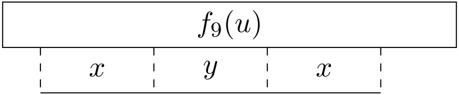

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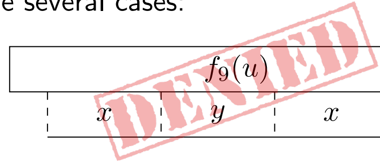

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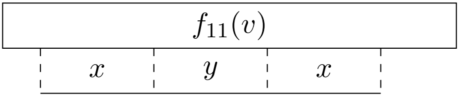

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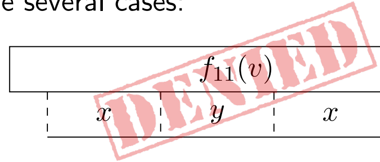

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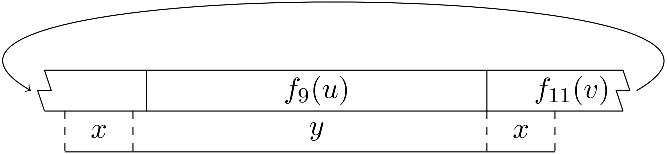
- Then have several cases:



Constructing circular $\frac{3}{2}^+$-free words on four letters

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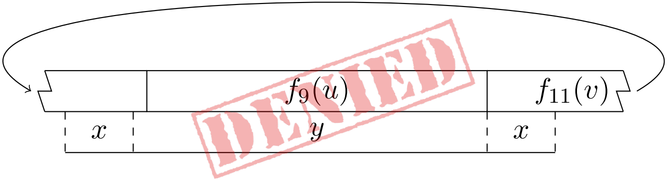

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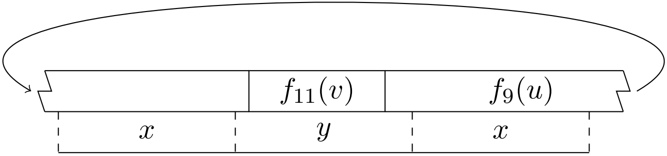

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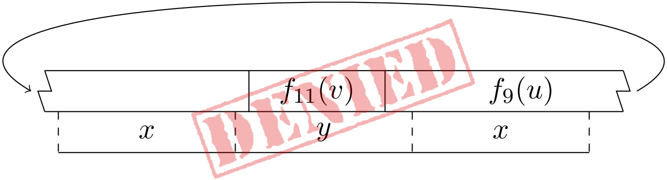

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$$\mathrm{CRT}(4) = \frac{3}{2}.$$

Plan

Background

Four letters

Five letters

Conclusion

Gorbunova's technique for larger alphabets

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- e.g. seven letter alphabet $\{0, 1, 2, 3, 4, 5, 6\}$.

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- Let

$$w = \boxed{05} \underbrace{\boxed{\quad u \quad}}_{\{0,1,2,3,4\}} \boxed{50} \underbrace{\boxed{\quad f(u) \quad}}_{\{2,3,4,5,6\}}$$

where u is $\frac{5}{4}^+$-free.

Constructing circular $\frac{4}{3}^{+}$-free words on five letters

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Idea:

Constructing circular $\frac{4}{3}^{+}$-free words on five letters

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- Get a $\frac{4}{3}^{+}$-free word on four letters and use a construction similar to Gorbunova's.

$$w = \boxed{04}\,\underbrace{\boxed{\quad u \quad}}_{\{0,1,2,3\}}\,\boxed{40}\,\underbrace{\boxed{\quad f(u) \quad}}_{\{1,2,3,4\}}$$

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Problem:

- $\mathrm{RT}(4) = \frac{7}{5}$, so every sufficiently long word on four letters has factors with exponent greater than $\frac{4}{3}$.

Constructing circular $\frac{4}{3}^+$-free words on five letters

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Solution:

Constructing circular $\frac{4}{3}^+$-free words on five letters

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- In his work showing that $\mathrm{RT}(4) = \frac{7}{5}$, Pansiot constructed words of every length where the only factors of exponent greater than $\frac{4}{3}$ look like

  0123 102132 0123

  up to permutation of the letters.

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$$w = \boxed{04}\ \underbrace{\boxed{\quad u \quad}}_{\substack{\{0,1,2,3\} \\ \text{(and a few 4's)}}}\ \boxed{40}\ \underbrace{\boxed{\quad f(u)^R \quad}}_{\substack{\{1,2,3,4\} \\ \text{(and a few 0's)}}}$$

Therefore,

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$$\mathrm{CRT}(5) = \frac{4}{3}.$$

Plan

Background

Four letters

Five letters

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We now know that

$$\mathrm{CRT}(k) = \begin{cases} \frac{5}{2} & \text{if } k = 2; \\ 2 & \text{if } k = 3; \\ \frac{\lceil k/2 \rceil + 1}{\lceil k/2 \rceil} & \text{if } k \geq 4. \end{cases}$$

Something to think about...

Conjecture

For all $k \geq 4$,

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Also think about attending the minisymposium on Open Problems in Combinatorics on Words on Thursday afternoon.