# The repetition threshold for binary rich words

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Turku, Finland
June 21, 2023

# PLAN

WORDS AND REPETITIONS

RICH WORDS

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  - `clementine` – square-free
- Question: Is there an infinite square-free word over a finite alphabet? What is the smallest such alphabet?

# THE ORIGIN OF COMBINATORICS ON WORDS

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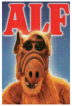

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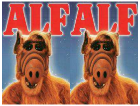

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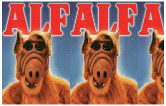

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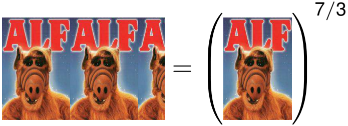

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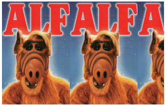
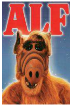



$=$ ( 

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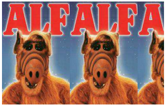
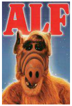



$=$ ( 

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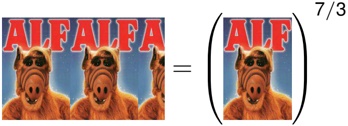


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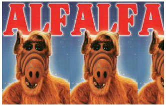
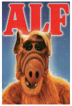



$=$ ( 

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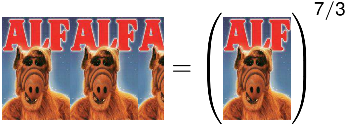


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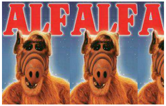


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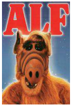


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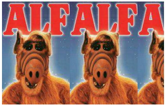
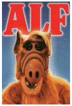



$= \Bigg($  $\Bigg)^{7/3}$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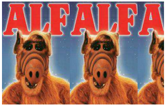
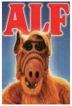



$=$ ( 

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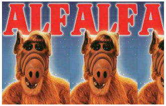


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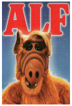


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- The repetition threshold for a set of words $L$ is the smallest critical exponent among all infinite words in $L$.
  - The repetition threshold for binary words is 2.

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- In particular, $w$ contains every finite factor of the Thue-Morse word.

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- The Thue-Morse word contains nothing “larger” than a square; it has critical exponent 2.
- Hence, the repetition threshold for binary words is 2.
- If an infinite binary word has critical exponent less than $7/3$, then it looks like the Thue-Morse word.

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- An infinite word is called rich if all of its finite factors are rich.

# Repetitions in rich words

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- In particular, what is the repetition threshold for binary rich words?
- That is, what is the smallest critical exponent among all infinite binary rich words?

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- They proved (by backtracking) that the repetition threshold is at least 2.7

# Baranwal and Shallit's construction

Define morphisms $f$ and $h$ by

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\begin{aligned}
f(\mathtt{0}) &= \mathtt{0} \\
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- The proof was completed using the automatic theorem proving software `Walnut`.

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  - Unfortunately, this is not the case!
  - Fortunately, it is not much worse than this.

# ANOTHER STRUCTURE THEOREM

Every infinite binary rich word with critical exponent less than $14/5$ looks like either $u = f(h^\omega(\texttt{0}))$ or $v = f(g(h^\omega(\texttt{0})))$.

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| $f(\texttt{0}) = \texttt{0}$ | $g(\texttt{0}) = \texttt{011}$ | $h(\texttt{0}) = \texttt{01}$ |
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Theorem (Currie, Mol, and Rampersad 2020): Let $w$ be an infinite binary rich word with critical exponent less than $14/5$. Then $w$ has a suffix of the form $f(h^n(w_n))$ or $f(g(h^n(w_n)))$ for all $n \geq 1$.

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- It suffices to show that both $u$ and $v$ are rich and have critical exponent $2 + \sqrt{2}/2$.
- Baranwal and Shallit handled $u$.
- We handle $v$.
- Our proof technique can also be applied to $u$, providing an alternate proof of Baranwal and Shallit's result.

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- Therefore, both $u$ and $v$ are rich!

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- Both $u$ and $v$ are complementary symmetric Rote words; they are rich and have critical exponent $2 + \sqrt{2}/2$.
- We conclude that the repetition threshold for binary rich words is $2 + \sqrt{2}/2$.

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- Dejean's Theorem: The repetition threshold for all words on $k$ letters is given by

$$\mathrm{RT}(k) = \begin{cases} 2, & \text{if } k = 2; \\ 7/4, & \text{if } k = 3; \\ 7/5, & \text{if } k = 4; \\ k/(k-1), & \text{if } k \geq 5. \end{cases}$$

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  - Do similar structure theorems hold over larger alphabets?
- During this trip: Some progress on the ternary case with Currie and Peltomäki.

Thank you!