On Connectivity of Orientations of Graphs

Lucas Mol

THOMPSON RIVERS UNIVERSITY

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Question: What proportion of the connectivity of a graph can be preserved under orientation?

Theorem (Nash-Williams, 1960): Every $2k$-edge-connected graph has a strongly k-arc-connected orientation.

Conjecture (Thomassen, 1989): For every positive integer k, there exists a smallest positive integer $f(k)$ such that every $f(k)$-connected graph has a strongly k-connected orientation.

Local vertex connectivity:

$$\kappa(u, v) = \text{max \# of pairwise internally disjoint } uv\text{-paths}$$
$$= \text{min size of a } uv\text{-cut (if } u \text{ and } v \text{ are not adjacent)}$$

Local edge connectivity:

$$\lambda(u, v) = \text{max \# of pairwise edge disjoint } uv\text{-paths}$$
$$= \text{min size of a } uv\text{-edge cut}$$

CONNECTIVITY

	Vertex version	Edge version
Global	$\kappa(G) = \min \kappa(u, v)$	$\lambda(G) = \min \lambda(u, v)$
Average	$\overline{\kappa}(G) = \text{average}\, \kappa(u, v)$	$\overline{\lambda}(G) = \text{average}\, \lambda(u, v)$

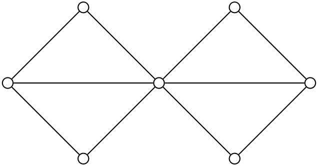
Maximum connectivity across all orientations

	Vertex version	Edge version
Global	$\kappa_{\rightarrow}(G)$	$\lambda_{\rightarrow}(G)$
Average	$\overline{\kappa}_{\rightarrow}(G)$	$\overline{\lambda}_{\rightarrow}(G)$

PROPORTION OF CONNECTIVITY THAT CAN BE PRESERVED UNDER ORIENTATION

	Vertex version	Edge version
Global	$\frac{\kappa_{\rightarrow}(G)}{\kappa(G)}$	$\frac{\lambda_{\rightarrow}(G)}{\lambda(G)}$
Average	$\frac{\overline{\kappa}_{\rightarrow}(G)}{\overline{\kappa}(G)}$	$\frac{\overline{\lambda}_{\rightarrow}(G)}{\overline{\lambda}(G)}$

AN EXAMPLE



$$\overline{\kappa}(G) = \frac{9\cdot1+10\cdot2+2\cdot3}{21} = \frac{5}{3}$$

$$\overline{\kappa}_{\rightarrow}(G) = \frac{40\cdot1+2\cdot2}{42} = \frac{22}{21}$$

$$\overline{\lambda}(G) = \frac{18\cdot2+3\cdot3}{21} = \frac{15}{7}$$

$$\overline{\lambda}_{\rightarrow}(G) = \frac{39\cdot1+3\cdot2}{42} = \frac{15}{14}$$

$$\frac{\overline{\kappa}_{\rightarrow}(G)}{\overline{\kappa}(G)} = \frac{22}{35}$$

$$\frac{\overline{\lambda}_{\rightarrow}(G)}{\overline{\lambda}(G)} = \frac{1}{2}$$

Edge Version

Theorem (Nash-Williams, 1960): $\lambda_{\rightarrow}(G) \geq \lfloor \lambda(G)/2 \rfloor$.

In fact, Nash-Williams proved the following stronger statement.

Theorem: Every graph G has an orientation D such that

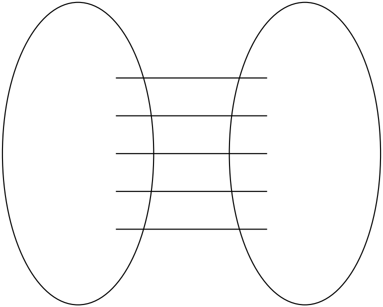
$$\lambda_D(u, v) \geq \lfloor \lambda_G(u, v)/2 \rfloor$$

for all ordered pairs (u, v).

- This gives:

Edge Version

For an upper bound, note that $\lambda_D(u, v) + \lambda_D(v, u) \leq \lambda_G(u, v)$.



Vertex Version

Conjecture (Thomassen, 1989): For every positive integer k, there exists a smallest positive integer $f(k)$ such that every $f(k)$-connected graph has a strongly k-connected orientation.

- If so, then what does $f(k)$ grow like?

Question: Is there some positive constant c such that

$$\frac{\overline{\kappa}_{\rightarrow}(G)}{\overline{\kappa}(G)} \geq c$$

for every nonempty graph G?

Vertex Version

For an upper bound, note that

$$\kappa_D(u, v) \leq \kappa_G(u, v)$$

for all ordered pairs (u, v).

- This gives:

This is asymptotically best possible.

SUMMARY OF GENERAL BOUNDS

Edge Version:

$$\frac{1}{2} - \frac{1}{2\overline{\lambda}(G)} \leq \frac{\overline{\lambda}_{\rightarrow}(G)}{\overline{\lambda}(G)} \leq \frac{1}{2}$$

Vertex Version:

$$c \overset{?}{\leq} \frac{\overline{\kappa}_{\rightarrow}(G)}{\overline{\kappa}(G)} \leq 1$$

TREES

Theorem (Henning and Oellermann, 2004): For every nontrivial tree T, we have

$$\frac{2}{9} < \frac{\overline{\kappa}_{\rightarrow}(T)}{\overline{\kappa}(T)} \leq \frac{1}{2},$$

and both bounds are asymptotically tight.

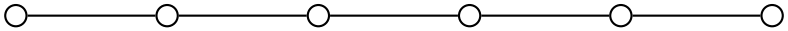
Question: Could it be that

$$\frac{2}{9} < \frac{\overline{\kappa}_{\rightarrow}(G)}{\overline{\kappa}(G)}$$

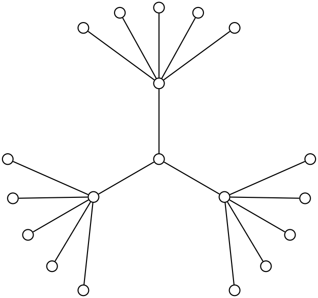
for every graph G?

Extremal trees

Upper Bound:



Lower Bound:



Other Graph Classes

The following are due to Casablanca, Dankelmann, Goddard, Mol, and Oellermann (2021).

Theorem: Every minimally 2-connected graph G satisfies

$$\frac{4}{9} < \frac{\overline{\kappa}_{\rightarrow}(G)}{\overline{\kappa}(G)} < \frac{5}{8}.$$

Theorem: Every maximal outerplanar graph G satisfies

$$\frac{1}{2} \leq \frac{\overline{\kappa}_{\rightarrow}(G)}{\overline{\kappa}(G)} < \frac{3}{4}.$$

Open problems

Question: Is there some positive constant c such that

$$\frac{\overline{\kappa}_{\rightarrow}(G)}{\overline{\kappa}(G)} \geq c$$

for every nonempty graph G?

Question: If G is 2-connected, then is every optimal orientation of G strong?

Thank you!