Avoiding Additive Powers in Words

Lucas Mol

THOMPSON RIVERS UNIVERSITY

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"The three spheres continued to dance in my dream, a patternless, never-repeating dance. Yet, in the depths of my mind, the dance did possess a rhythm."

–from *The Three-Body Problem* by Cixin Liu

PLAN

SQUARES AND SQUARE-FREE WORDS

ABELIAN AND ADDITIVE SQUARES

OUTLOOK

A PROOF SKETCH

Alphabets and words

- An *alphabet* is a finite set of letters, treated simply as symbols, e.g.,
  - $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}\}$ (the English alphabet)
  - $\{\mathtt{0}, \mathtt{1}\}$ (the binary alphabet)
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- Which patterns can be avoided, and which patterns must inevitably occur?

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  - `apple` – not square-free
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- One can define cubes, 4th powers, etc. in a similar manner.

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h^3(\mathtt{0}) &= \mathtt{012021012102} \\
h^4(\mathtt{0}) &= \mathtt{012021012102012021020121} \\
&\vdots \\
h^\omega(\mathtt{0}) &= \mathtt{012021012102012021020121}\cdots
\end{aligned}$$

THE ORIGIN OF COMBINATORICS ON WORDS

Theorem: $h^{\omega}(\mathtt{0}) = \mathtt{012021012102012021020121}\ldots$ is square-free.

Axel Thue (1863-1922)

Plan

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- Unlike regular squares, they are NOT avoidable over three letters.
- Theorem (Keränen 1992): Abelian squares are avoidable over four letters.
- The word $\sigma^{\omega}(\mathtt{0})$ avoids abelian squares, where

$\sigma(0) = \mathtt{0120232123203231301020103101213121021232021013010203212320231210212320232132303132120}$

$\sigma(1) = \mathtt{1231303230310302012131210212320232132303132120121310323031302321323031303203010203231}$

$\sigma(2) = \mathtt{2302010301021013123202321323031303203010203231232021030102013032030102010310121310302}$

$\sigma(3) = \mathtt{3013121012132120230313032030102010310121310302303132101213120103101213121021232021013}$

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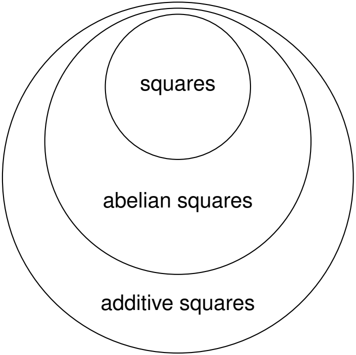

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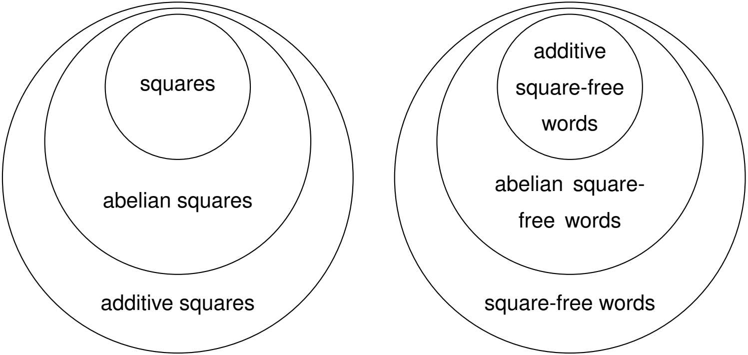

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- The word

$$h^{\omega}(\mathtt{0}) = \mathtt{031430110343430310110110314 3}\cdots$$

avoids additive cubes, where

$$\begin{aligned} h(\mathtt{0}) &= \mathtt{03} \\ h(\mathtt{1}) &= \mathtt{43} \\ h(\mathtt{3}) &= \mathtt{1} \\ h(\mathtt{4}) &= \mathtt{01} \end{aligned}$$

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- The algorithm is easy to implement.
- It is efficient enough to work in practice!

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- Show that these seed words cannot look "too different" from additive powers – there are only finitely many possible *templates* for these seed words.
- Enumerate all short words in $h^\omega(\mathtt{0})$, and check to see if they match any of the templates.

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- Define f by

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- We can establish this fact by simply running our algorithm.
- In fact, $g(f^{\omega}(0))$ is additive 4th power-free, where

$$g(\mathtt{0}) = \mathtt{0001001110010001100011}$$
$$g(\mathtt{1}) = \mathtt{0001001110011101100011}$$
$$g(\mathtt{2}) = \mathtt{0111001110011101100011}$$

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Abelian and Additive Squares

Outlook

A Proof Sketch

OUTLOOK

We still don't have a construction of additive square-free words over a finite subset of $\mathbb{Z}$.

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Hopefully this method will prove useful!

Plan

Squares and Square-Free Words

Abelian and Additive Squares

Outlook

A Proof Sketch

The theorem statement

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Let's find out by sketching the proof.

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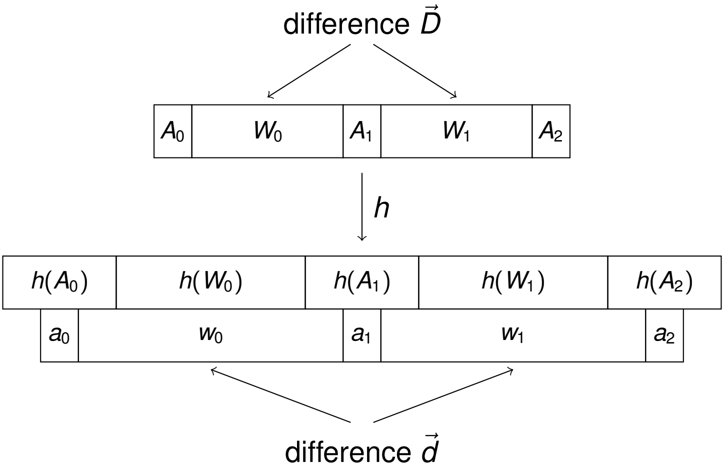
- An instance of $[\varepsilon, \varepsilon, \varepsilon, \vec{0}]$ is an additive square.
- An instance of $[0, 1, 0, [1, 3]^T]$ is "not too far" from an additive square.

PARENTS

Every long-enough instance of a template must have come from an instance of another template – a *parent*.



The First Two Conditions

- Condition 1: For all letters x,
  - the length of $h(x)$ is given by $ax + b$ for some $a, b \in \mathbb{Z}$, and
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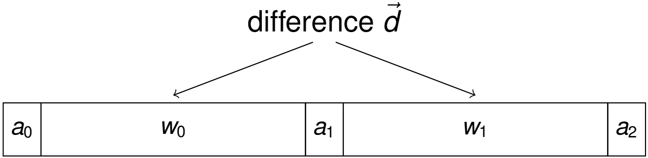
- Then $\vec{\sigma}(h(W)) = M_h \vec{\sigma}(W)$.
- Condition 2: M_h is invertible, so that

$$\vec{\sigma}(W) = M_h^{-1} \vec{\sigma}(h(W)).$$

FINDING PARENTS

These first two conditions allow us to find all possible parents of a given template.



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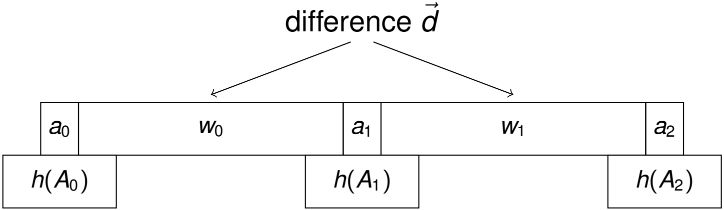

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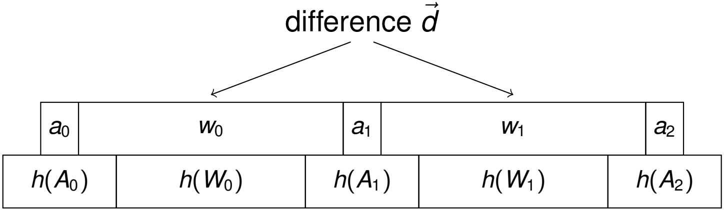

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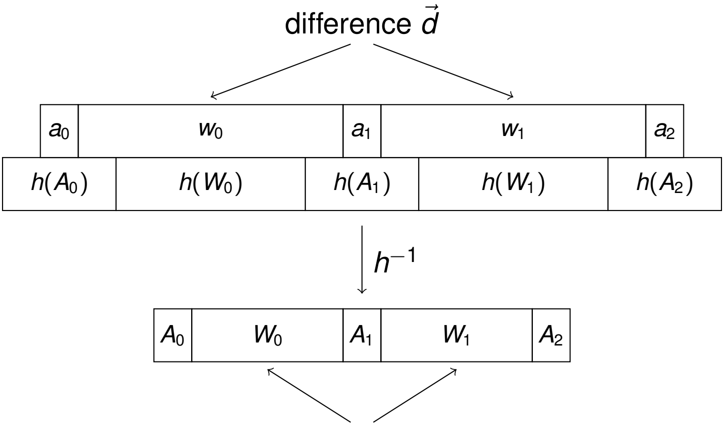

difference $\vec{D}$ determined by $\vec{d}$ and the choice/position of the $h(A_i)$'s

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- Condition 3: All eigenvalues of M_h are larger than 1 in absolute value.

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- So taking preimages makes words shorter!
- So if $h^\omega(0)$ contains an instance of a template t, then $h^\omega(0)$ contains a *short* instance of some ancestor of t.

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  - If so, then $h^\omega(0)$ contains an additive square.
  - If not, then $h^\omega(0)$ is additive square-free!

Thank you!