Avoiding Additive Powers in Words

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Plan

Power Avoidance

Decision Algorithms

Outlook

Alphabets and words

- An alphabet is a finite set of letters, e.g.,
  - $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}\}$ (the English alphabet)
  - $\{\mathtt{0}, \mathtt{1}\}$ (the binary alphabet)
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- Which patterns can be avoided, and which patterns must inevitably occur?

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- Question: Are squares avoidable over some finite alphabet?
  - That is, are there arbitrarily long square-free words over some finite alphabet?

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h^4(\mathtt{0}) &= \mathtt{012021012102012021020121} \\
&\vdots \\
h^\omega(\mathtt{0}) &= \mathtt{012021012102012021020121}\cdots
\end{aligned}
$$

The origin of combinatorics on words

Theorem: $h^{\omega}(0) = \mathtt{012021012102012021020121}\ldots$ is square-free.

Axel Thue (1863-1922)

Abelian and Additive Squares

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- Theorem (Keränen 1992): $\sigma^\omega(\mathtt{0})$ avoids abelian squares, where

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  - We don't know!

Some Progress

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Theorem (Rao and Rosenfeld 2018): Additive squares are avoidable over a finite subset of $\mathbb{Z}^2$.

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Theorem (Currie, Mol, Rampersad, and Shallit 2021+): A more efficient algorithm for additive powers (with stronger conditions on h).

- The algorithm is easy to implement.
- It is efficient enough to work in practice, even for “long” substitutions!

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- Define f by

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  is additive 4th-power-free.
- We can establish this fact by simply running our algorithm.
- In fact, $g(f^{\omega}(0))$ is additive 4th-power-free, where

$$g(\texttt{0}) = \texttt{0001001110010001100011}$$
$$g(\texttt{1}) = \texttt{0001001110011101100011}$$
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- Enumerate all short words in $h^\omega(\texttt{0})$, and check to see if they match any of the templates.
- Our stronger conditions on h allow us to greatly reduce the number of templates that need to be checked.

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Hopefully this method will prove useful!


Thank you!

Proof Sketch

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Let's find out by sketching the proof.

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- A word w is an instance of this template if

$$w = a_0 w_0 a_1 w_1 a_2 \quad \text{and} \quad \vec{\sigma}(w_1) - \vec{\sigma}(w_0) = \vec{d}.$$

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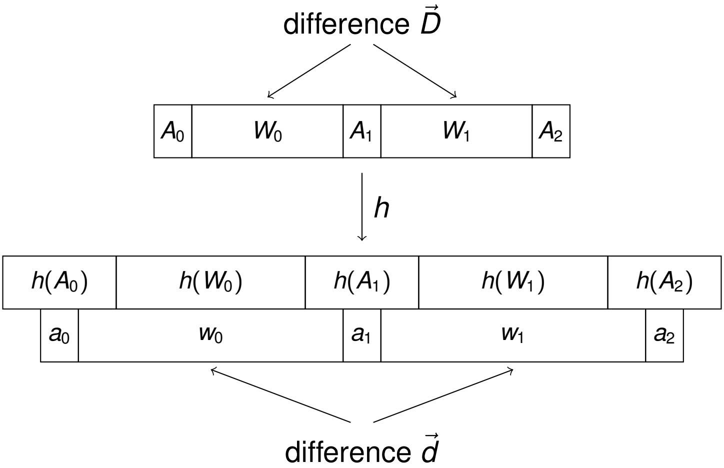
- An instance of $[\varepsilon, \varepsilon, \varepsilon, \vec{0}]$ is an additive square!
- An instance of $[0, 1, 0, [1, 3]^T]$ is "not too far" from an additive square.

PARENTS

Every long-enough instance of a template must have come from an instance of another template – a parent.



The First Two Conditions

- Condition 1: For all letters x,
  - the length of $h(x)$ is given by $ax + b$ for some $a, b \in \mathbb{Z}$, and
  - the sum of $h(x)$ is given by $cx + d$ for some $c, d \in \mathbb{Z}$.

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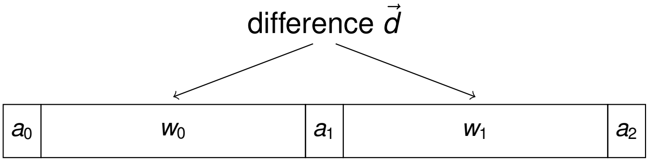
$$\vec{\sigma}(W) = M_h^{-1} \vec{\sigma}(h(W)).$$

Finding Parents

These first two conditions allow us to find all possible parents of a given template.



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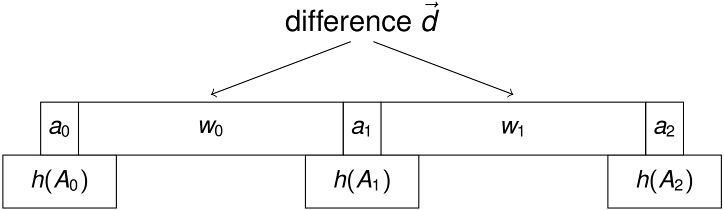

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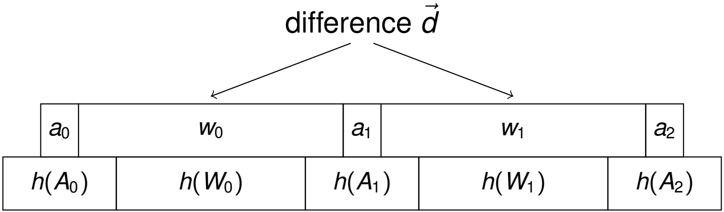

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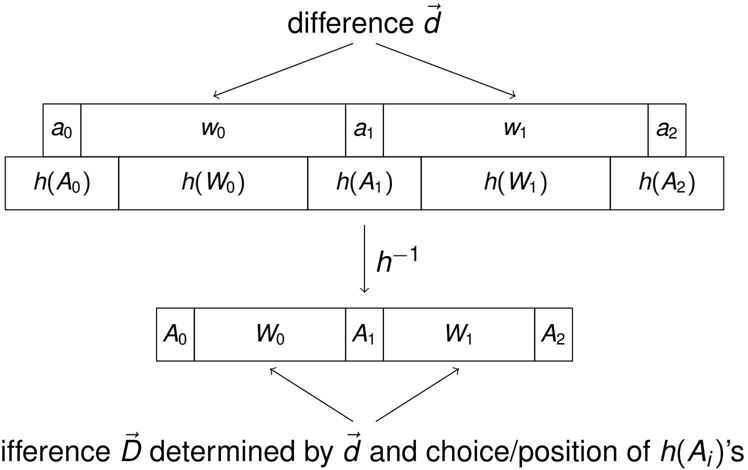

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  - If so, then $h^\omega(0)$ contains an additive square.
  - If not, then $h^\omega(0)$ is additive square-free!