Avoiding Additive Powers in Words

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> –from *The Three-Body Problem* by Cixin Liu

PLAN

SQUARES AND SQUARE-FREE WORDS

ABELIAN AND ADDITIVE SQUARES

ALPHABETS AND WORDS

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    - $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}\}$ (the English alphabet)
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- Which patterns can be avoided, and which patterns must inevitably occur?

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    - `apple` – not square-free
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- One can define cubes, 4th powers, etc. in a similar manner.

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- Over an alphabet of size three, say $\{0, 1, 2\}$?

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THE ORIGIN OF COMBINATORICS ON WORDS

Theorem: $h^{\omega}(\mathtt{0}) = \mathtt{012021012102012021020121}\ldots$ is square-free.

Axel Thue (1863-1922)

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- Unlike regular squares, they are NOT avoidable over three letters.
- Theorem (Keränen 1992): Abelian squares are avoidable over four letters.
- The word $\sigma^{\omega}(\mathtt{0})$ avoids abelian squares, where

$\sigma(\mathtt{0}) = \mathtt{0120232123203231301020103101213121021232021013010203212320231210212320232132303132120}$
$\sigma(\mathtt{1}) = \mathtt{1231303230310302012131210212320232132303132120121310323031302321323031303203010203231}$
$\sigma(\mathtt{2}) = \mathtt{2302010301021013123202321323031303203010203231232021030102013032030102010310121310302}$
$\sigma(\mathtt{3}) = \mathtt{3013121012132120230313032030102010310121310302303132101213120103101213121021232021013}$

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- The word $h^\omega(\texttt{0})$ avoids additive cubes, where

$$\begin{aligned} h(\texttt{0}) &= \texttt{03} \\ h(\texttt{1}) &= \texttt{43} \\ h(\texttt{3}) &= \texttt{1} \\ h(\texttt{4}) &= \texttt{01} \end{aligned}$$

DECISION ALGORITHMS

- Theorem (Currie and Rampersad 2012): There is an algorithm which decides, under certain conditions on h, whether $h^\omega(\mathtt{0})$ contains abelian squares (cubes, 4th powers, etc.)

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- Theorem (Currie, Mol, Rampersad, and Shallit 2021+): Stronger conditions on h, more efficient algorithm for additive powers.

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- Show that these seed words cannot look "too different" from additive powers – there are only finitely many possible *templates* for these seed words.
- Enumerate all short words in $h^{\omega}(\texttt{0})$, and check to see if they match any of the templates.

EXAMPLE

Define f by

$$\begin{aligned} f(\texttt{0}) &= \texttt{001} \\ f(\texttt{1}) &= \texttt{012} \\ f(\texttt{2}) &= \texttt{212} \end{aligned}$$

Then

$$f^{\omega}(\texttt{0}) = \texttt{0010010120010010120010122 12}\cdots$$

is additive 4th power-free.

- Our theorem shows that this fact can be established by a finite computer check.
- This word is also (regular) cube-free.

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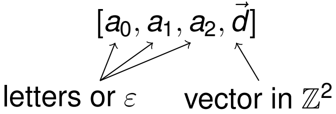
Hopefully this method will prove useful!

Thank you!

TEMPLATES

- Let $\vec{\sigma}(w)$ denote the vector $\begin{bmatrix}\text{length of } w \\ \text{sum of } w\end{bmatrix}$.
- A *template* (for additive squares) is a 4-tuple



$$[a_0, a_1, a_2, \vec{d}]$$

letters or ε

vector in $\mathbb{Z}^2$

- A word w is an *instance* of $[a_0, a_1, a_2, d]$ if

$$w = a_0 w_0 a_1 w_1 a_2$$

and

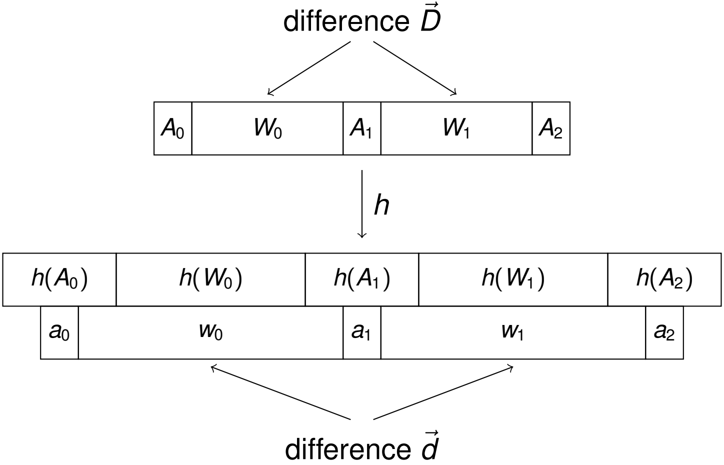
$$\vec{\sigma}(w_1) - \vec{\sigma}(w_0) = \vec{d}.$$

- If $w = x\tilde{x}$ is an additive square, then w is an instance of

$$[\varepsilon, \varepsilon, \varepsilon, \vec{0}]$$

PARENTS

We say that template $T = [A_0, A_1, A_2, \vec{D}]$ is a *parent* of template $t = [a_0, a_1, a_2, \vec{d}]$ if applying h to an instance of T gives an instance of t.



difference $\vec{D}$

A_0 W_0 A_1 W_1 A_2

h

$h(A_0)$ $h(W_0)$ $h(A_1)$ $h(W_1)$ $h(A_2)$

a_0 w_0 a_1 w_1 a_2

difference $\vec{d}$

THE FIRST TWO CONDITIONS

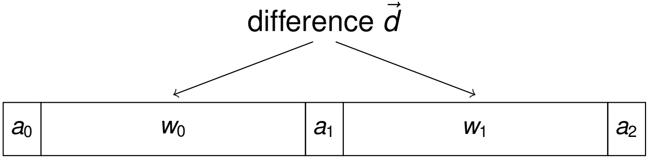
- Condition 1: For all letters x, the length and sum of $h(x)$ are *linear* functions of x, that is
  - the length of $h(x)$ is $ax + b$
  - the sum of $h(x)$ is $cx + d$

  for some $a, b, c, d \in \mathbb{Z}$.
- Record this in the matrix $M_h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- Lemma: $M_h \vec{\sigma}(W) = \vec{\sigma}(h(W))$
- Condition 2: M_h is invertible, so that

$$\vec{\sigma}(W) = M_h^{-1} \vec{\sigma}(h(W)).$$

FINDING PARENTS

These first two conditions allow us to find all possible parents of a given template $t = [a_0, a_1, a_2, \vec{d}]$.



difference $\vec{d}$

a_0 | w_0 | a_1 | w_1 | a_2

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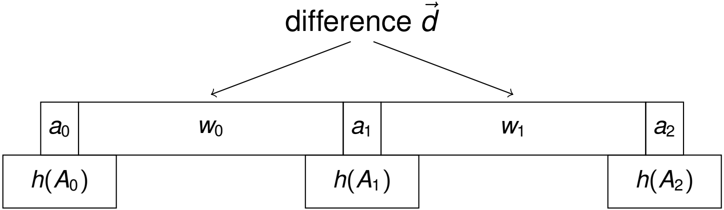

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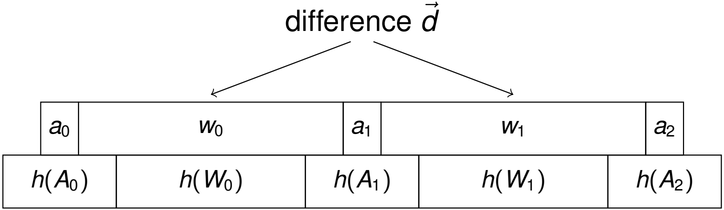

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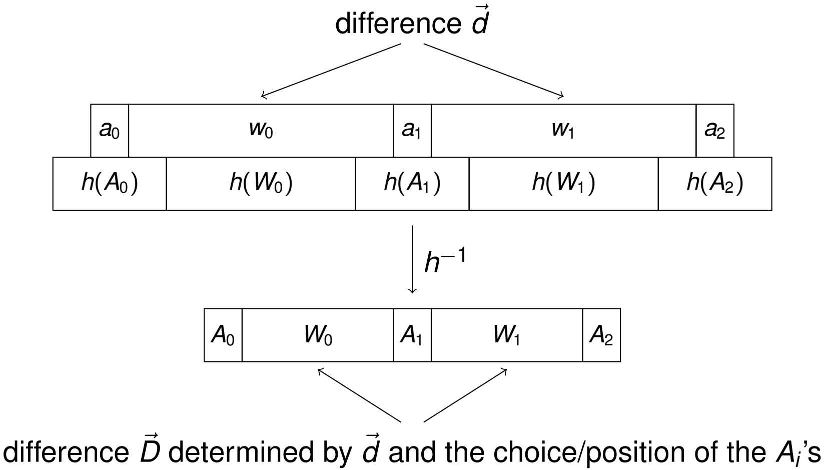

difference $\vec{d}$

a_0 w_0 a_1 w_1 a_2

$h(A_0)$ $h(W_0)$ $h(A_1)$ $h(W_1)$ $h(A_2)$

h^{-1}

A_0 W_0 A_1 W_1 A_2

difference $\vec{D}$ determined by $\vec{d}$ and the choice/position of the A_i's

THE THIRD CONDITION

- Now that we can compute the parents of a given template t, we want to compute the set of all *ancestors* of t (parents, grandparents, great-grandparents, etc.)
- How do we know that this set is finite?
- We need a condition on h which guarantees that for any ancestor $T = [A_0, A_1, A_2, \vec{D}]$ of t, the difference $\vec{D}$ is not too large.
- Essentially, applying M_h^{-1} repeatedly cannot make the differences larger and larger.
- Condition 3: All eigenvalues of M_h are larger than 1 in absolute value.

THE LAST CONDITION

- Condition 4: For all letters x, the length of $h(x)$ is at least 2.
- So taking preimages makes words shorter!
- This guarantees that if $h^{\omega}(0)$ contains an instance of a template t, then $h^{\omega}(0)$ contains a *short* instance of some ancestor of t.

DESCRIPTION OF THE ALGORITHM

Suppose that *h* satisfies these four conditions.

- Consider the template $t = [\varepsilon, \varepsilon, \varepsilon, \vec{0}]$.
  - An instance of *t* is an additive square.
- We enumerate all ancestors of *t*.
  - This set is finite!
- If $h^\omega(0)$ contains an additive square, then it must contain a *short* instance of one of these ancestors.
  - This is our “seed word”.
- We enumerate all short factors of $h^\omega(0)$, and check to see if any of them is an instance of an ancestor of *t*.
  - If so, then $h^\omega(0)$ contains an additive square.
  - If not, then $h^\omega(0)$ is additive square-free!