Avoiding Additive Powers in Words

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“The three spheres continued to dance in my dream, a patternless, never-repeating dance. Yet, in the depths of my mind, the dance did possess a rhythm.”

–from *The Three-Body Problem* by Cixin Liu

PLAN

SQUARES AND SQUARE-FREE WORDS

ABELIAN AND ADDITIVE SQUARES

OUTLOOK

A PROOF SKETCH

Alphabets and words

- An *alphabet* is a finite set of letters, treated simply as symbols, e.g.,
  - $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}\}$ (the English alphabet)
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- Which patterns can be avoided, and which patterns must inevitably occur?

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  - `apple` – not square-free
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- One can define cubes, 4th powers, etc. in a similar manner.

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$$\begin{aligned}
h(\mathtt{0}) &= \mathtt{012} \\
h^2(\mathtt{0}) &= \mathtt{012021} \\
h^3(\mathtt{0}) &= \mathtt{012021012102} \\
h^4(\mathtt{0}) &= \mathtt{012021012102012021020121} \\
&\vdots \\
h^\omega(\mathtt{0}) &= \mathtt{012021012102012021020121}\ldots
\end{aligned}$$

THE ORIGIN OF COMBINATORICS ON WORDS

Theorem: $h^{\omega}(0) = \texttt{012021012102012021020121}\ldots$ is square-free.

Axel Thue (1863-1922)

Plan

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Abelian and Additive Squares

Outlook

A Proof Sketch

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- Unlike regular squares, they are NOT avoidable over three letters.
- Theorem (Keränen 1992): Abelian squares are avoidable over four letters.
- The word $\sigma^{\omega}(\mathtt{0})$ avoids abelian squares, where

$\sigma(0) = $ 0120232123203231301020103101213121021232021013010203212320231210212320232132303132120
$\sigma(1) = $ 1231303230310302012131210212320232132303132120121310323031302321323031303203010203231
$\sigma(2) = $ 2302010301021013123202321323031303203010203231232021030102013032030102010310121310302
$\sigma(3) = $ 3013121012132120230313032030102010310121310302303132101213120103101213121021232021013

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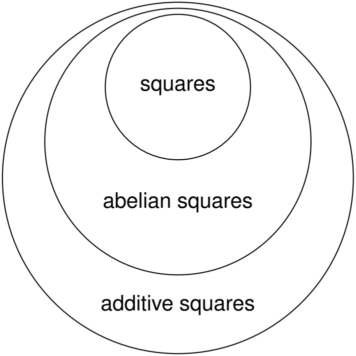

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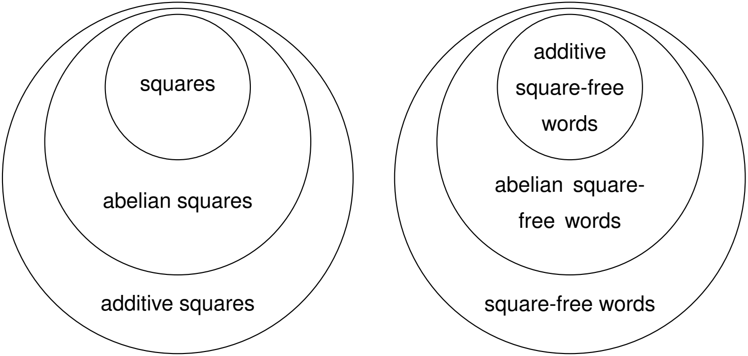

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- The word $h^{\omega}(\mathtt{0})$ avoids additive cubes, where

$$h(\mathtt{0}) = \mathtt{03}$$
$$h(\mathtt{1}) = \mathtt{43}$$
$$h(\mathtt{3}) = \mathtt{1}$$
$$h(\mathtt{4}) = \mathtt{01}$$

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- Theorem (Rao and Rosenfeld 2018): Weaker conditions on h, less efficient algorithm. Can be tweaked to handle abelian or additive powers.
- Theorem (Currie, Mol, Rampersad, and Shallit 2021+): Stronger conditions on h, more efficient algorithm for additive powers.

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- Show that these seed words cannot look "too different" from additive powers – there are only finitely many possible *templates* for these seed words.
- Enumerate all short words in $h^\omega(\texttt{0})$, and check to see if they match any of the templates.

Example

Define f by

$$f(\mathtt{0}) = \mathtt{001}$$
$$f(\mathtt{1}) = \mathtt{012}$$
$$f(\mathtt{2}) = \mathtt{212}$$

Then

$$f^{\omega}(\mathtt{0}) = \mathtt{001001012001001012001012212}\cdots$$

is additive 4th power-free.

- Our theorem shows that this fact can be established by a finite computer check.
- This word is also (regular) cube-free.

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Outlook

We still don't have a construction of additive square-free words over a finite subset of $\mathbb{Z}$.

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- The conditions of our theorem are fairly restrictive.
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Hopefully this method will prove useful!

Plan

Squares and Square-Free Words

Abelian and Additive Squares

Outlook

A Proof Sketch

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Let’s find out by sketching the proof.

TEMPLATES

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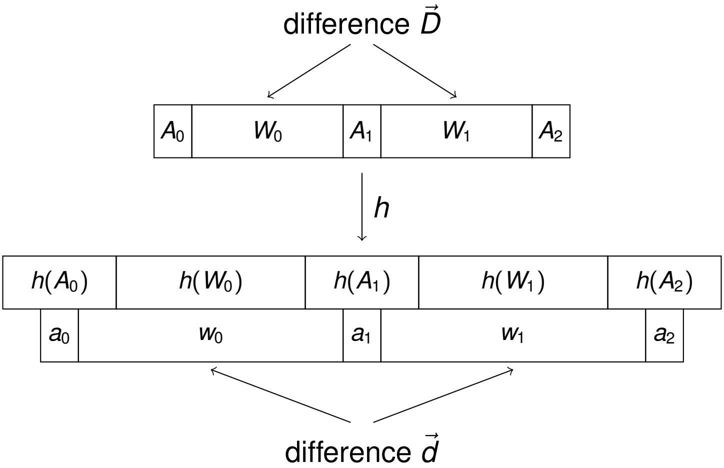
- An instance of $[\varepsilon, \varepsilon, \varepsilon, \vec{0}]$ is an additive square.
- An instance of $[0, 1, 0, [1, 3]^T]$ is “not too far” from an additive square.

PARENTS

Every long-enough instance of a template must have come from an instance of another template – a *parent*.



The First Two Conditions

- Condition 1: For all letters x,
  - the length of $h(x)$ is given by $ax + b$ for some $a, b \in \mathbb{Z}$, and
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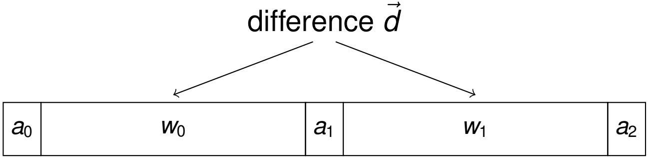
- Then $\vec{\sigma}(h(W)) = M_h \vec{\sigma}(W)$.
- Condition 2: M_h is invertible, so that

$$\vec{\sigma}(W) = M_h^{-1} \vec{\sigma}(h(W)).$$

Finding Parents

These first two conditions allow us to find all possible parents of a given template.



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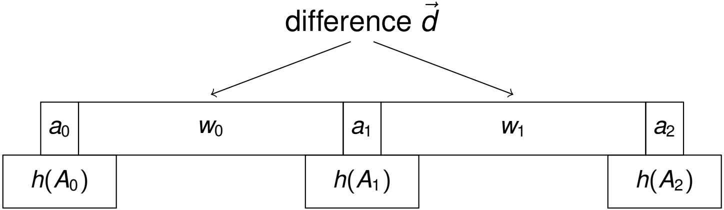

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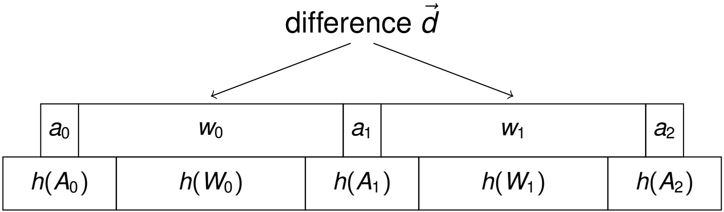

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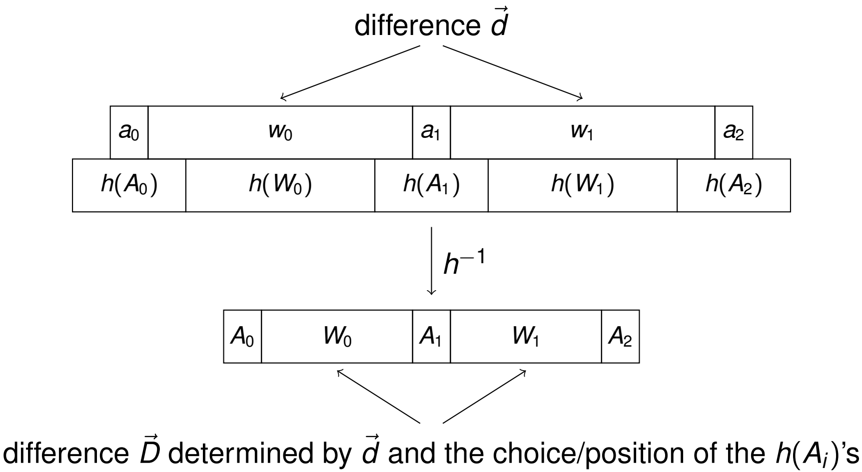

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- Condition 3: All eigenvalues of M_h are larger than 1 in absolute value.

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- So if $h^{\omega}(0)$ contains an instance of a template t, then $h^{\omega}(0)$ contains a *short* instance of some ancestor of t.

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Suppose that h satisfies these four conditions.

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  - If so, then $h^\omega(0)$ contains an additive square.
  - If not, then $h^\omega(0)$ is additive square-free!

Thank you!