Chapter 3 – Fluid Statics

3.1 Pressure

Consider a small cylinder of fluid at rest as shown in Figure 3.1. The cylinder has a length L and a cross-sectional area ΔA. Since the cylinder is at rest, the sum of the forces acting along the axis of the cylinder must be equal to zero. The pressure forces on the sides of the cylinder act perpendicular to the axis and therefore do not have any component acting along the cylinder axis. The only forces acting along this axis are the pressure force acting on the upper end of the cylinder, $p_1\Delta A$, the pressure force at the lower end of the cylinder is $p_2\Delta A$, and the weight of the volume of liquid in the cylinder, $W = \rho g \cdot Vol$.

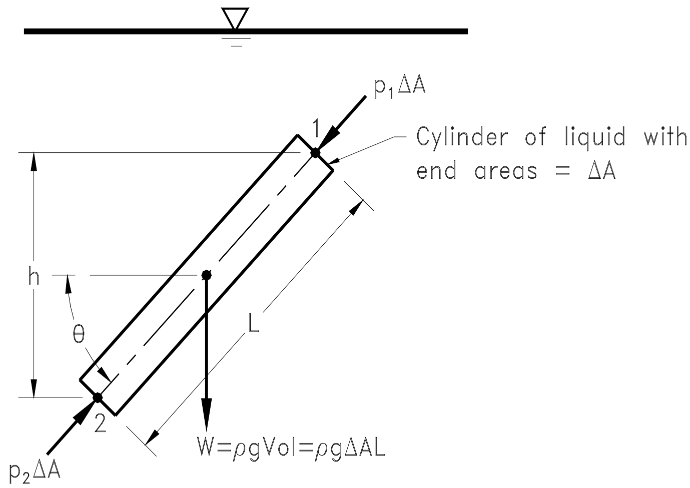


Figure 3.1 Pressure variation in a liquid

Applying equilibrium along the axis, we have

$$p_2\Delta A - p_1\Delta A - \rho g\Delta AL \cdot \sin\theta = 0$$

Dividing through by ΔA, and noting that $L \cdot \sin\theta = h$, yields

$$p_2 - p_1 = \rho g \cdot h \qquad (3.1)$$

It can then be stated that the change in pressure is equal to the weight density of the liquid multiplied by the change in depth. Note that if point 1 and point 2 are at the same elevation, then $h = 0$ and hence $p_2 = p_1$. In other words, the pressure within any continuous expanse of fluid is constant at a given elevation.

Pressure can be measured relative to a vacuum or relative to the atmospheric pressure. Pressure measured relative to a vacuum is known as **absolute pressure**. Pressure measured relative to atmospheric pressure is known as **gauge pressure**. Gauge pressures can be negative or positive as indicated in Figure 3.2. Negative gauge pressures indicate a pressure that is less than atmospheric; positive gauge pressures indicate a pressure that is greater than atmospheric pressure.

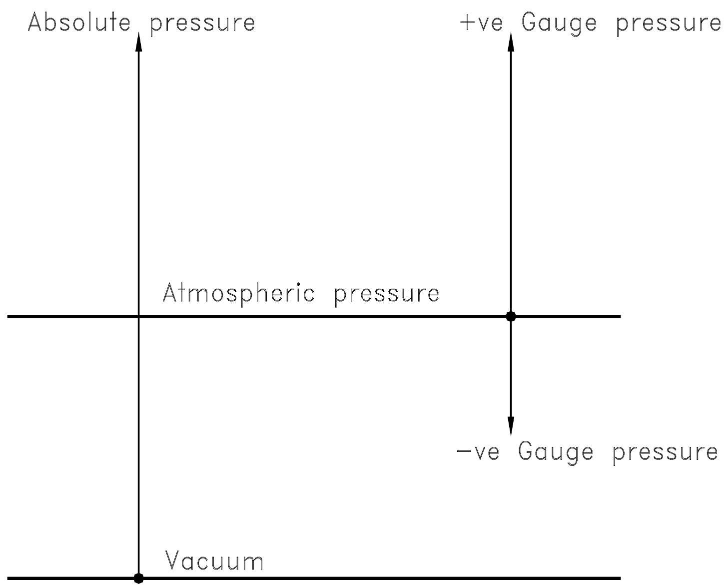


Figure 3.2 Absolute and gauge pressures.

The free surface of a liquid will have an absolute pressure $p = p_a$, where p_a is the atmospheric pressure. The absolute pressure at a depth h in the fluid will be

$$p = p_a + \rho g \cdot h \tag{3.2}$$

For a gauge pressure, p_a is taken as zero and the gauge pressure becomes

$$p = \rho g \cdot h \tag{3.3a}$$

Equation 3.3a can be solved for h to give

$$h = \frac{p}{\rho g} \tag{3.3b}$$

It is often convenient to measure the elevation of a point above a fixed datum. This elevation is given the letter z. Within a continuous expanse of fluid, the height z measured from a fixed datum plus the height h is constant, i.e.

$$\frac{p}{\rho g} + z = \text{constant} \tag{3.4}$$

This result is illustrated in Figure 3.3. The left side of equation 3.4 is known as the **piezometric head.**

Most pressures used in common types of engineering problems are gauge pressures. However, when vapor pressure is a consideration, absolute pressures must be examined.

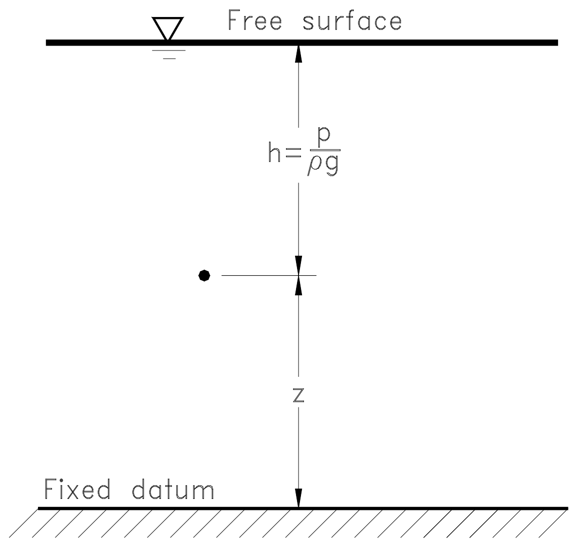


Figure 3.3 Piezometric head.

Hydraulic rams and hydraulic jacks make use of the fact that a pressure increase applied to a fluid is transmitted throughout the fluid. This principle is illustrated in Figure 3.4 which shows a simplified diagram of a hydraulic jack.

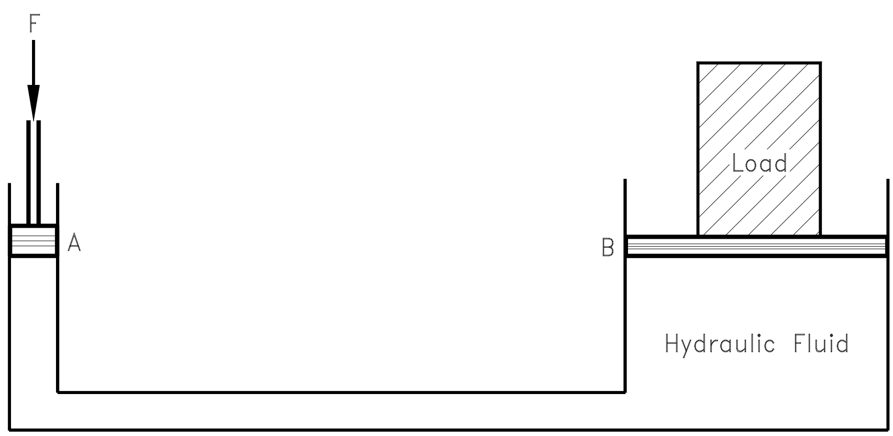


Figure 3.4 Hydraulic jack.

A force is applied to the small cylinder, A, which is transmitted into a pressure over the area of the small piston. The pressure is then transmitted throughout the hydraulic fluid. The pressure exerts a larger force ($F = p \cdot A$) on the large piston, B. It should be noted that, neglecting friction, the work done by the applied force acting on the small piston is the same as the work done by the large piston as it lifts the load.

Example 3.1

A hydraulic jack modeled after the one shown in Figure 3.4 is to lift a load of 10 kN. The diameter of the small piston is 20mm and the diameter of the large piston is 200 mm. Determine the force, F, required to lift the load.

Solution:

The pressure on the large piston that is required to lift the load is

$$p = \frac{F}{A} = \frac{10{,}000\,N}{\pi(100\,mm)^2} = 0.3183 \text{ MPa}$$

The force required on the smaller piston to raise the load is

$$F = p \cdot A = (0.3183 \text{ MPa})\ \pi\ (10 \text{ mm})^2 = \underline{\underline{100\ N}}.$$

3.2 Manometers

Manometers are devices used to measure relative pressure. Manometers can be of two types:

1.) Open manometers consist of a tube leading from the vessel with the far end of the tube is open to the atmosphere. The tube is bent and it contains a manometer fluid. Open manometers measure the pressure in a vessel (or pipe) relative to atmospheric pressure, that is the gauge pressure. See Figure 3.5a.

2.) Closed manometers are used to measure the relative pressure between two different vessels (or pipes). See Figure 3.5b.

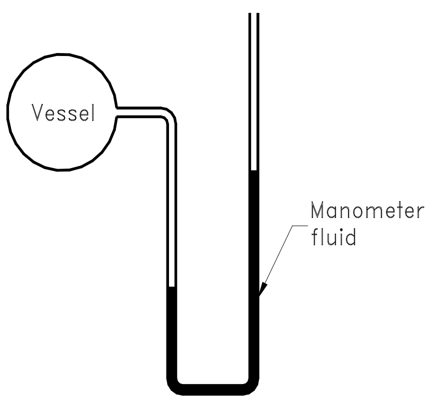


Figure 3.5a Open Manometer

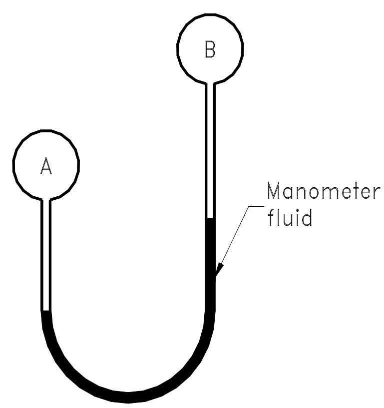


Figure 3.5b Closed Manometer

Pressure calculations for manometers are based on two basic principles.

1.) The pressure at a given elevation within a continuous expanse of the same fluid is constant.

2.) The pressure in a fluid increases with increasing depth and decreases with decreasing depth according to the relation $p = \rho g \cdot h$.

Manometer fluids should be immiscible with the vessel fluid and the contact surface between the fluids should be clearly visible. The density of the manometer fluid can be less than or greater than the density of the vessel fluid(s). The use of less dense manometer fluids results in a more sensitive instrument, but one that is limited in range. Manometer tubes can have an upward curved loop or a downward curved loop, to accommodate differing densities. Manometers can also have a series of loops. For very sensitive pressure measurements, the manometer tube may be given a shallow slope which has the effect of stretching out the liquid and facilitates the reading of finer measurements.

For open manometers, the design procedure is to start at the open end with a pressure $p_{gauge} = 0$, and apply the two principles working in towards the vessel. For closed manometers, one can start with the pressure in either vessel, (for example vessel B with the assumed pressure p_B) and work towards the other vessel. All that can be determined is the pressure difference, either $p_B - p_A$ or $p_A - p_B$.

Example 3.2

Figure 3.6 shows an open manometer. The manometer has an upward loop with Alcohol, relative density 0.84 and a downward loop with mercury, relative density of 13.56. The vessel, A, contains water at 10°C. Determine the gauge pressure in the vessel.

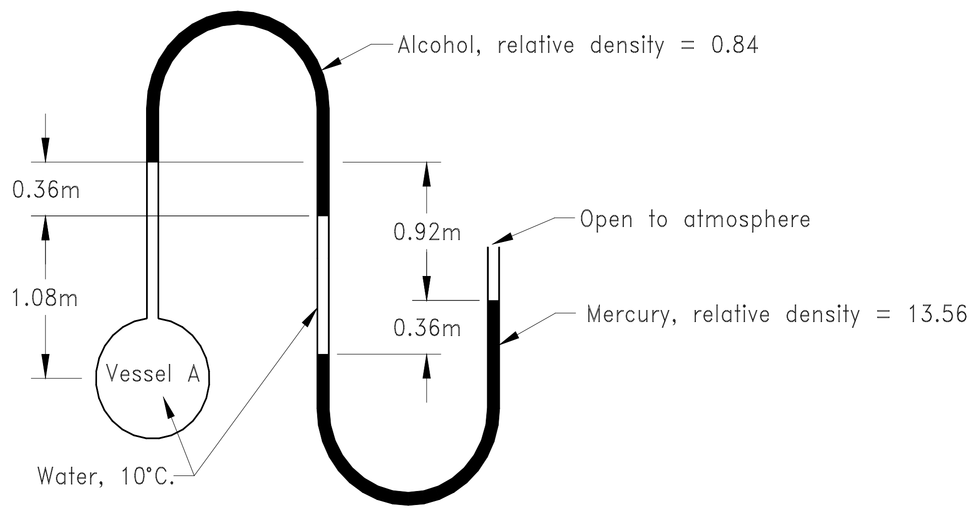


Figure 3.6

Solution:

Starting at the open end of the manometer,

$$p_A = [(13560\ \text{kg/m}^3)(0.36\text{m}) - (999\ \text{kg/m}^3)(0.92\text{m}) - (840\ \text{kg/m}^3)(0.36\text{m})$$

$$+ (999\ \text{kg/m}^3)(1.44\text{m})](9.81\ \text{m/s}^2) = \underline{50.02\ \text{kPa}}$$

3.3 Pressure on Submerged Plane Surfaces

When a plane surface is immersed in a liquid, the pressure exerts a distributed force over the entire surface. For most engineering problems, it is desirable to determine the magnitude of the resultant force due to the pressure and to determine the location where this resultant force can be considered to act.

Consider the diagram in Figure 3.7.

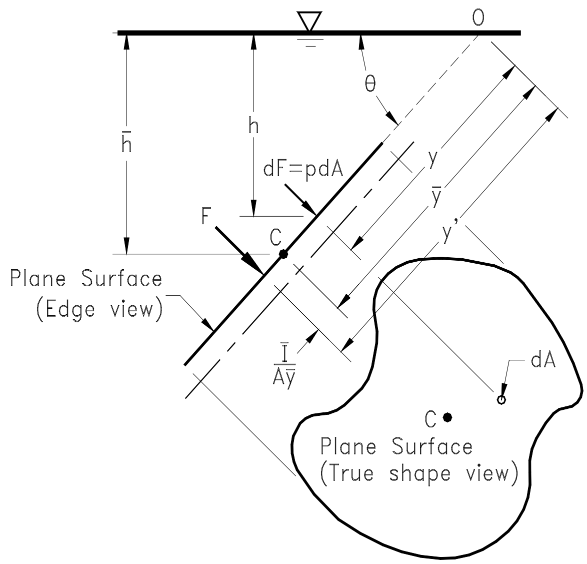


Figure 3.7 Resultant Force Acting on a Submerged Surface

The pressure force increases with depth according to the relation $p = \rho g \cdot h$, and always acts perpendicular to the plane surface. The net force acting on a small element of area dA is $dF = p \cdot dA = \rho g \cdot h \cdot dA = \rho g \cdot y \sin\theta \cdot dA$, where y is the distance measured down the incline of the plane from the point O on the free surface of the fluid. If we sum up all the elements of force, dF, we get the total resultant force. The summing up can be performed by taking the integral of both sides and excluding constants from the integration.

$$F = \int dF = \rho g \sin\theta \int_A y\, dA \qquad (3.5)$$

The right side integral is known as the first moment of an area. The same integral appears in the definition of the location of the centroid of an area. This centroid location is given as

$$\bar{y} = \frac{\int_A y\, dA}{A} \qquad (3.6)$$

From equation 3.6, the integral $\int_A y\, dA = A\,\bar{y}$.

Substituting into equation 3.5 gives

$$F = \rho g\, A\, \bar{y} \sin\theta \qquad (3.7)$$

If we designate $\bar{h} = \bar{y}\sin\theta$ as the vertical depth to the centroid of the area, the equation for the resultant force becomes

$$F = \rho g\, A \bar{h} \qquad (3.8)$$

To locate this resultant, we will have to apply a principle of engineering mechanics known as the principle of moments or Varignon's theorem. Varignon's theorem states that the sum of the moments of the components of a force about a point is equal to the moment of the resultant force about the same point. We will take moments about the point O on the free surface.

The moment of an element of force dF is $y \cdot dF = \rho g \cdot y^2 \sin\theta \cdot dA$. Summing up an applying Varignon's theorem yields

$$M_O = F \cdot y' = \int y \cdot dF = \rho g \sin\theta \int_A y^2 dA \qquad (3.9)$$

The integral on the right side commonly occurs in mechanics and represents the moment of inertia of the area about point O and is given the designation I_O.

Substituting equation 3.8 into 3.9 and solving for y', the distance measured from the free surface down the slope to the location where the resultant force F can be considered to act, is

$$y' = \frac{\rho g \sin\theta\, I_O}{\rho g\, A\bar{y} \sin\theta} = \frac{I_O}{A\bar{y}} \qquad (3.10)$$

It is usually more convenient to locate the resultant relative to the centroid of the area and in terms of the centroidal moment of inertia designated by $\bar{I}$. To do this we must apply what is known as the parallel axis theorem. This theorem is usually written as

$$I_O = \bar{I} + Ad^2 \qquad (3.11)$$

where d is the distance between the centroid and the point O. In this case the distance d is equal to $\bar{y}$. Substituting equation 3.11 into 3.10 gives

$$y' = \frac{\bar{I}}{A\bar{y}} + \frac{A\bar{y}^2}{A\bar{y}}$$

or

$$y' = \frac{\bar{I}}{A\bar{y}} + \bar{y} \qquad (3.12)$$

The term $\frac{\bar{I}}{A\bar{y}}$ represents the distance measured down the slope from the centroid to the point where the resultant force can be considered to act. Note that this implies that the resultant always acts below the centroid of the area. Note also that for vertical surfaces, $\bar{y} = \bar{h}$. Centroid distances and centroidal moments of inertia are given in Table 3.1.

Table 3.1 Centroids and Moments of Inertia of Common Shapes

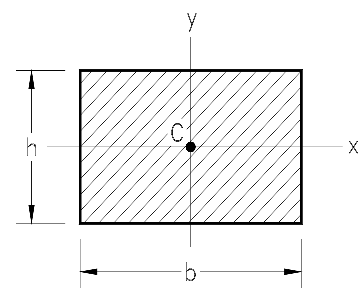
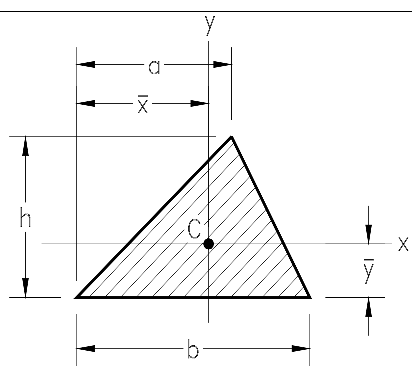
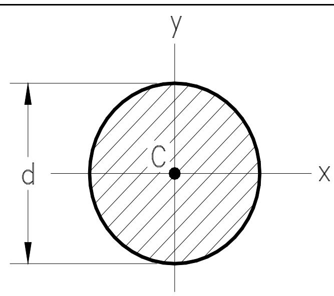
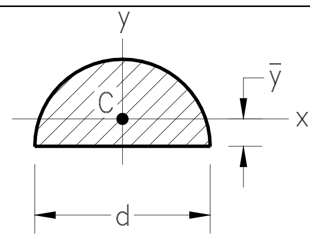
Shape	Area	Centroid	Centroidal Moment of Inertia
 Rectangle	$b \cdot h$	$\bar{x} = \frac{b}{2}$ $\bar{y} = \frac{h}{2}$	$\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_y = \frac{hb^3}{12}$
 Triangle	$\frac{b \cdot h}{2}$	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$\bar{I}_x = \frac{bh^3}{36}$ $\bar{I}_y = \frac{bh}{36}(a^2 - ab + b^2)$
 Circle	$\frac{\pi d^2}{4}$	$\bar{x} = \frac{d}{2}$ $\bar{y} = \frac{d}{2}$	$\bar{I}_x = \bar{I}_y = \frac{\pi d^4}{64}$
 Semi-circle	$\frac{\pi d^2}{8}$	$\bar{y} = \frac{2d}{3\pi}$	$\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)\frac{d^4}{16}$ $\bar{I}_y = \frac{\pi d^4}{128}$

Table 3.1 Centroids and Moments of Inertia of Common Shapes, Cont'd.

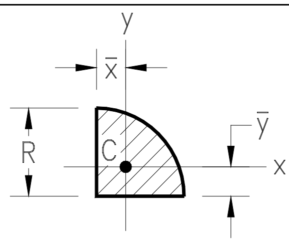
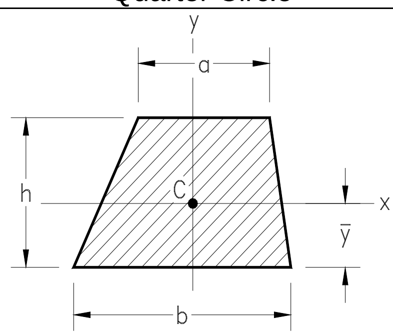
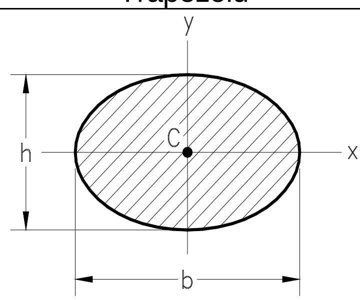
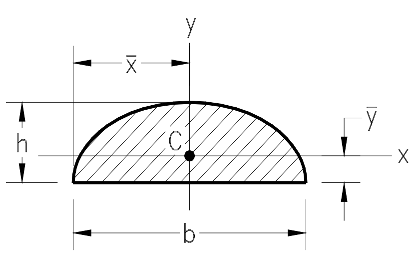
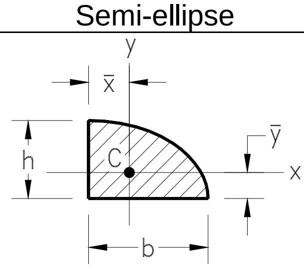
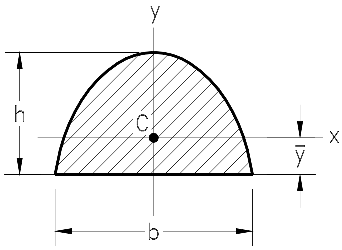
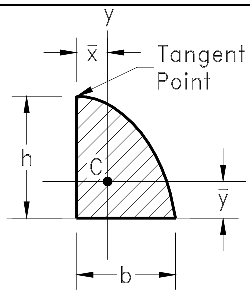
Shape	Area	Centroid	Centroidal Moment of Inertia
 Quarter Circle	$\dfrac{\pi R^2}{4}$	$\bar{x} = \bar{y} = \dfrac{4R}{3\pi}$	$\bar{I}_x = \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) R^4$
 Trapezoid	$\dfrac{h(a+b)}{2}$	$\bar{y} = \dfrac{h(2a+b)}{3(a+b)}$	$\bar{I}_x = \dfrac{h^3(a^2+4ab+b^2)}{36(a+b)}$
 Ellipse	$\dfrac{\pi(b \cdot h)}{4}$	$\bar{x} = \dfrac{b}{2}$ $\bar{y} = \dfrac{h}{2}$	$\bar{I}_x = \dfrac{\pi b h^3}{64}$ $\bar{I}_y = \dfrac{\pi h b^3}{64}$
 Semi-ellipse	$\dfrac{\pi(b \cdot h)}{4}$	$\bar{y} = \dfrac{4h}{3\pi}$	$\bar{I}_x = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) b h^3$ $\bar{I}_y = \dfrac{\pi h b^3}{128}$
 Quarter Ellipse	$\dfrac{\pi(b \cdot h)}{4}$	$\bar{x} = \dfrac{4b}{3\pi}$ $\bar{y} = \dfrac{4h}{3\pi}$	$\bar{I}_x = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) b h^3$ $\bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) h b^3$

Table 3.1 Centroids and Moments of Inertia of Common Shapes

Shape	Area	Centroid	Centroidal Moment of Inertia
 Parabola	$\frac{2b\cdot h}{3}$	$\bar{y} = \frac{2h}{5}$	$\bar{I}_x = \frac{8bh^3}{175}$ $\bar{I}_y = \frac{hb^3}{30}$
 Half Parabola	$\frac{2b\cdot h}{3}$	$\bar{x} = \frac{3b}{8}$ $\bar{y} = \frac{2h}{5}$	$\bar{I}_x = \frac{8bh^3}{175}$ $\bar{I}_y = \frac{19hb^3}{480}$

Example 3.3

The diagram below shows a circular gate in the side of a water (10°C) reservoir. The gate has a diameter of 0.80 m and is hinged along its upper edge as shown. The top of the gate is located 2.4 m below the surface of the water. The gate is built into a wall of the reservoir which has a slope of 60° from the horizontal as shown.

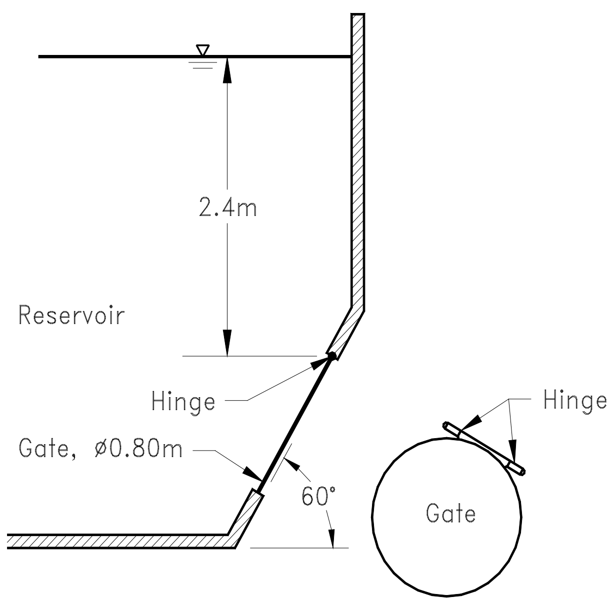


Figure 3.8

Determine:

a.) the resultant of the pressure forces acting on the gate.

b.) the location of the resultant pressure force with respect to the hinge.

c.) the moment of this force about the hinge.

Solution:

a.) $F = \rho g A \bar{h}$
$= (999\ kg/m^3)(9.81\ m/s^2)\pi(0.4\ m)^2\,[2.4\ m + (0.4m)(\sin 60°)]$

$F = \underline{\underline{13.53\ kN}}$

b.) F acts at $\frac{\bar{I}}{A\bar{y}}$ measured down the slope from the center of the gate.

$$\bar{y} = \frac{\bar{h}}{\sin\theta} = \frac{2.4\,m + (0.4\,m)(\sin 60°)}{\sin 60°} = 3.171\,m$$

$$\frac{\bar{I}}{A\bar{y}} = \frac{\pi d^4}{64}\cdot\frac{4}{\pi d^2}\cdot\frac{1}{\bar{y}} = \frac{d^2}{16\,\bar{y}} = \frac{(0.80m)^2}{16\,(3.171\,m)} = 0.01261\,m$$

Therefore the location of F measured down the slope from the hinge is 0.40 m + 0.01261 m = $\underline{\underline{0.4126\ m}}$.

c.) $M = F{\cdot}d = (13.53\ kN)(0.4126\ m) = \underline{\underline{5.58\ kN{\cdot}m}}$

3.4 Resultant Pressure Force on a Curved Surface

The resultant pressure force acting on a curved surface is determined by finding the horizontal and vertical components separately. Although the theory presented here applies to surfaces that curve in two directions, we will limit the discussion to surfaces curved in one direction only.

The horizontal component of the resultant force, F_x, is the same force that would occur on a vertical plane that is a horizontal projection of the curved surface.

The curved surface a_1,b_1,c_1,d_1 shown in Figure 3.9 is projected onto the vertical plane defined by a_2,b_2,c_2,d_2. The resultant horizontal force that would act on this vertical plane is given by Equation 3.8 as $F_x = \rho g A \bar{h}$. Here the area, A, and the depth to the centroid of the area, $\bar{h}$, both refer to the vertical projection. This horizontal component will act at a depth below the surface equal to

$$y' = \frac{\bar{I}}{A\bar{h}} + \bar{h} \qquad (3.12a)$$

Again the area, A, the moment of inertia, $\bar{I}$, and the depth to the centroid of the area, $\bar{h}$, all refer to the surface defined by the vertical projection of the curved surface.

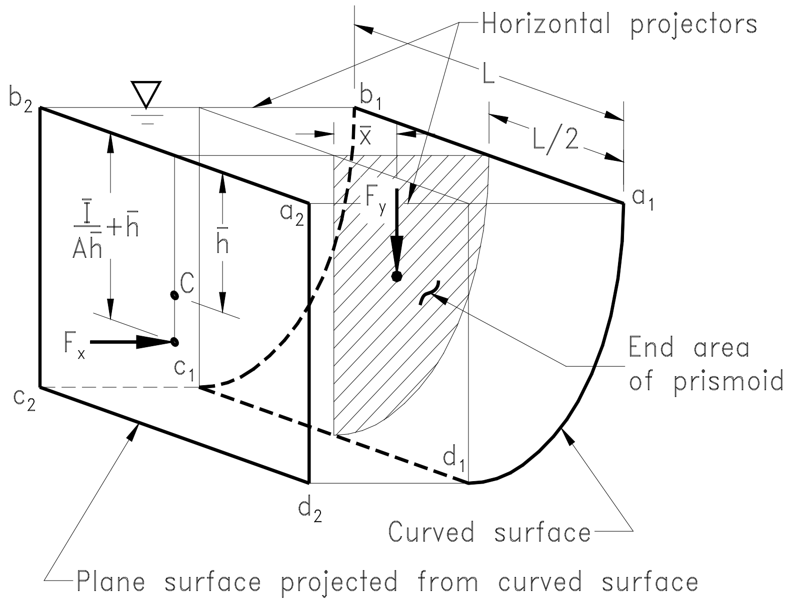


Figure 3.9 Components of the Resultant Pressure Force.

The vertical component of the resultant force acting on a curved surface is equal to the weight of the water lying above the curved surface, $F_y = \rho g \cdot Vol$. This vertical resultant force acts through the center of gravity of the volume of water that lies above the curved surface. For surfaces that are curved in one direction only and of a uniform length, the volume can be expressed as the product of the end area of the end area of the prismoid, A_p, and the length. Also for this situation, the center of gravity is located half way along the length and passes through the centroid of the end area of the prismoidal volume thus defined. Therefore the vertical component is $F_y = \rho g \cdot A_p L$

In some cases, the curved surface may have liquid beneath it, but not above it. In these situations, an imaginary volume of liquid equivalent to the volume above the curved surface and extending to the elevation of the liquid surface must be used. The resulting F_y component acts up, not down.

The resultant force F acts at the point where F_x and F_y intersect.

$$F = \sqrt{F_x^2 + F_y^2} \tag{3.13}$$

F acts at the angle

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right). \tag{3.14}$$

Example 3.4

Figure 3.10 shows a section through a small dam, 8.0m long. The back face of the dam is parabolic. If the depth of the water behind the dam is 4.0m, determine the horizontal and vertical components of the resultant of the pressure force and locate the point on the diagram where these forces can be considered to act. Also determine the magnitude and direction of the total resultant pressure force.

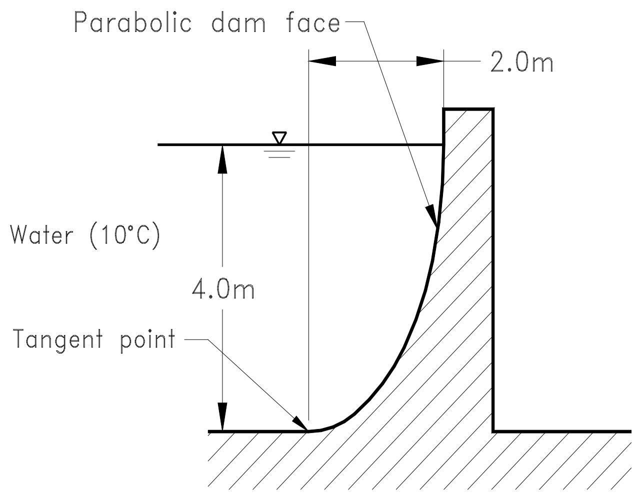


Figure 3.10

Solution:

The horizontal component $F_x = \rho g\, A\bar{h}$ = (999 kg/m^3)(9.81 m/s^2)(4.0 m)(8.0 m)(2.0 m)

F_x = 627.2 kN →

F_x acts at $y' = \frac{\bar{I}}{A\bar{h}} + \bar{h} = \frac{bh^3}{12} \cdot \frac{1}{bh} \cdot \frac{2}{h} + \frac{h}{2} = \frac{2h}{3} = \frac{2(4.0\,m)}{3} = \underline{\underline{2.667\ m}}$ below the free surface.

$F_y = \rho g \cdot A_p L$ = (999 kg/m^3)(9.81 m/s^2)(2/3)(2.0 m)(4.0 m)(8.0 m) = 418.1 kN ↓

F_y acts at $\bar{x} = \frac{3b}{8} = \frac{3(2.0\,m)}{8} = \underline{\underline{0.75\ m.}}$

Therefore $F = \sqrt{F_x^2 + F_y^2} = \sqrt{(627.2\,kN)^2 + (418.1\,kN)^2} = \underline{\underline{753.8\ kN}}$.

F acts at $\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{418.1\,kN}{627.2\,kN}\right) = \underline{\underline{33.69°}}$.

The results are summarized in Figure 3.11.

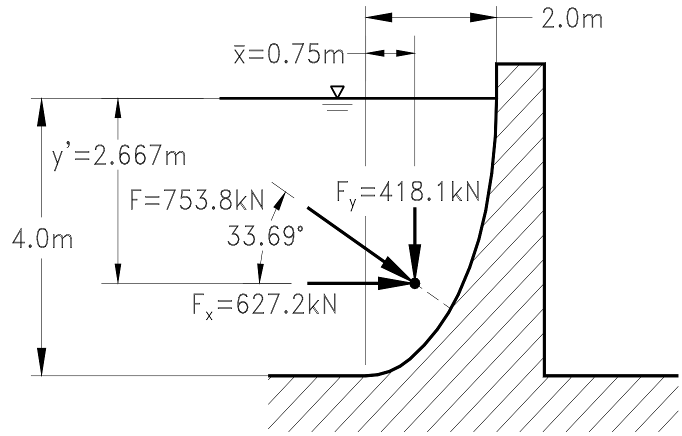


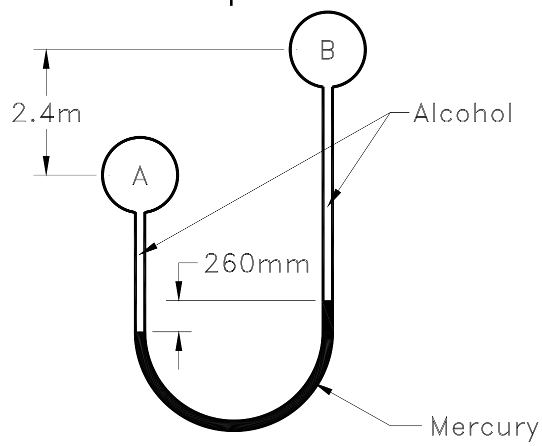
Figure 3.11

Problems – Chapter 3

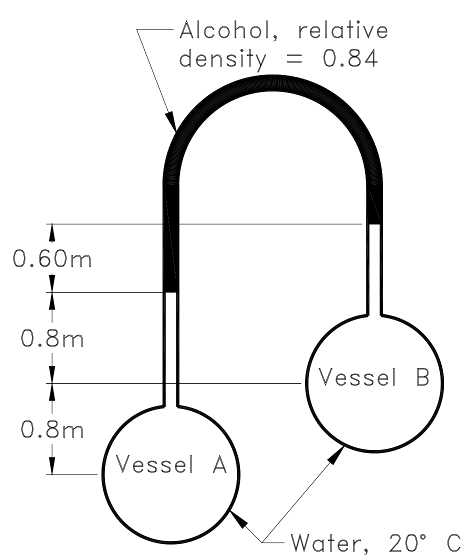
3.1) Two small vessels are connected to a U-tube manometer containing mercury (relative density = 13.56) and the connecting tubes are filled with alcohol (relative density = 0.82). The vessel at the higher pressure is 2.4 m lower in elevation than the other vessel.

a.) Determine the pressure difference between the two vessels when the difference in the levels of the mercury menisci is 260 mm.

b.) Determine the difference in the piezometric head.

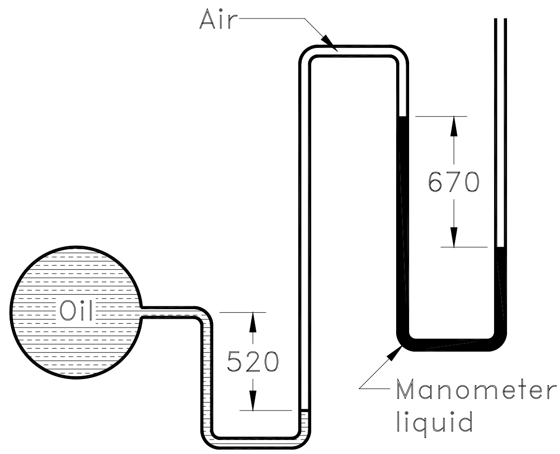


3.2) The diagram shows an inverted U-tube manometer with colored alcohol (relative density = 0.84). This type of manometer is useful for measuring small pressure differences. Determine the pressure difference between vessels A and B, both of which contain water at 20° C, if the manometer reading is 0.60 m.

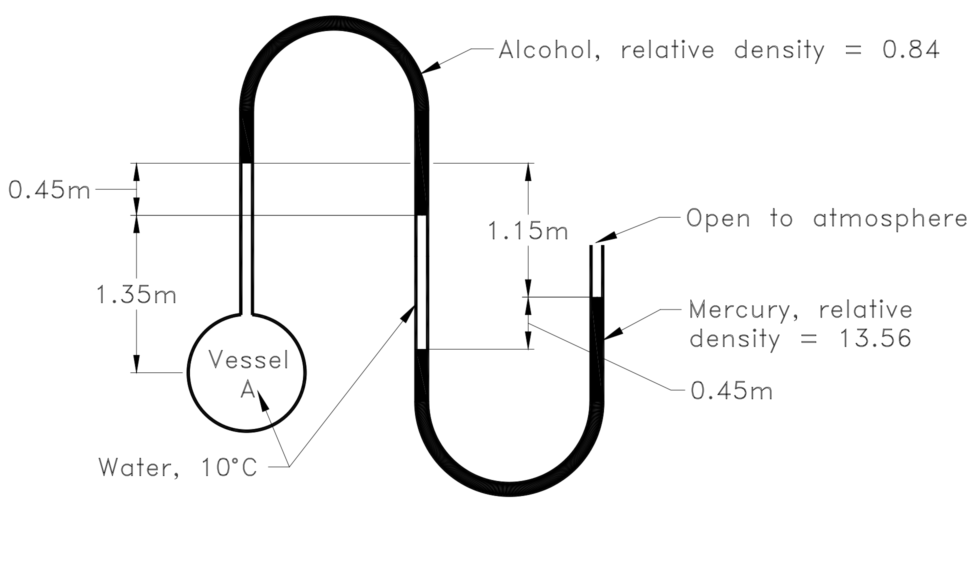


3.3) An open manometer is installed to measure the pressure in a pipe carrying oil (specific gravity = 0.82). If the manometer liquid is carbon tetrachloride (specific gravity = 1.60), determine the pressure in the pipe. Give the solution as a pressure in kPa, and, as a pressure head in meters of water column height.

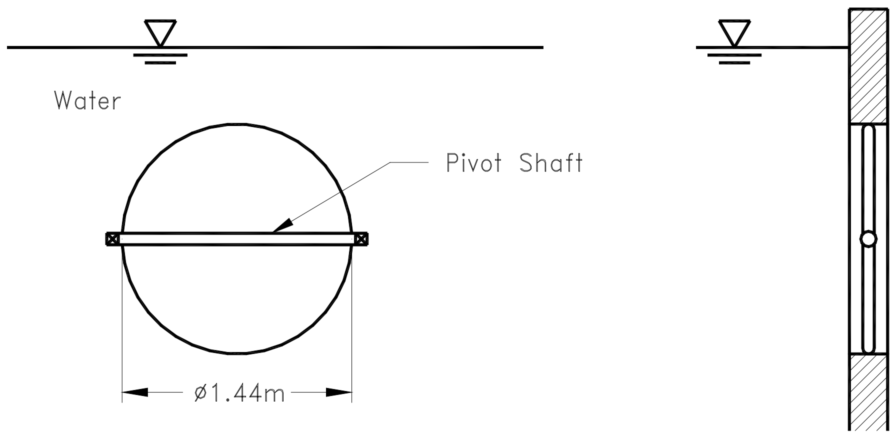
Hint: The weight of the column of air can be considered negligible.



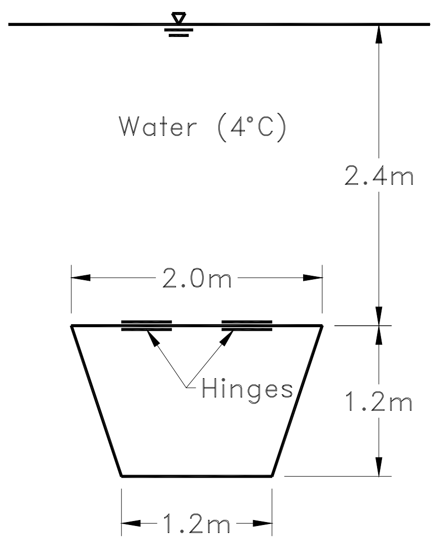
3.4) The diagram below shows a double U-tube open manometer. Determine the gauge pressure in vessel 'A'.



3.5) A 1.44 m diameter circular opening in the side of a vertical tank wall is closed by a circular gate that just fits the opening. The gate is pivoted on a shaft passing through its horizontal diameter. Show that the moment or torque on the shaft that is required to hold the disk vertical when the water level is above the top of the gate is independent of the water depth. Calculate the magnitude of this moment.



3.6) A trapezoidal gate sits in a vertical tank wall with its top edge located 2.4 m below the surface as shown in the diagram below. What is the moment of the resultant pressure force acting on the gate about the hinges located on the top edge?

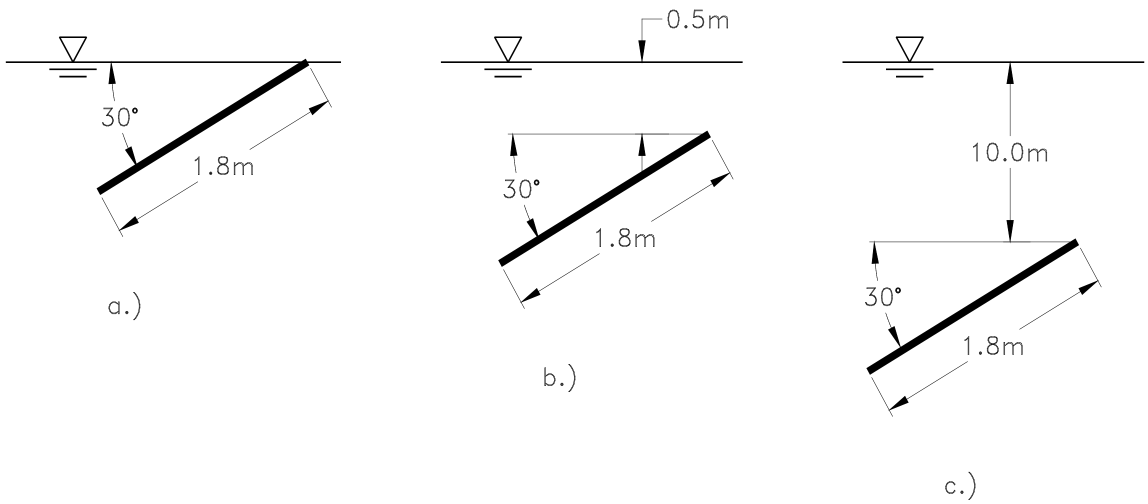


3.7) A 1.2 m by 1.8 m rectangular plate is submerged in water. The plate makes an angle of 30° with the horizontal. Calculate the magnitude of the net force acting on one face and the position of the center of pressure when the top plane is:

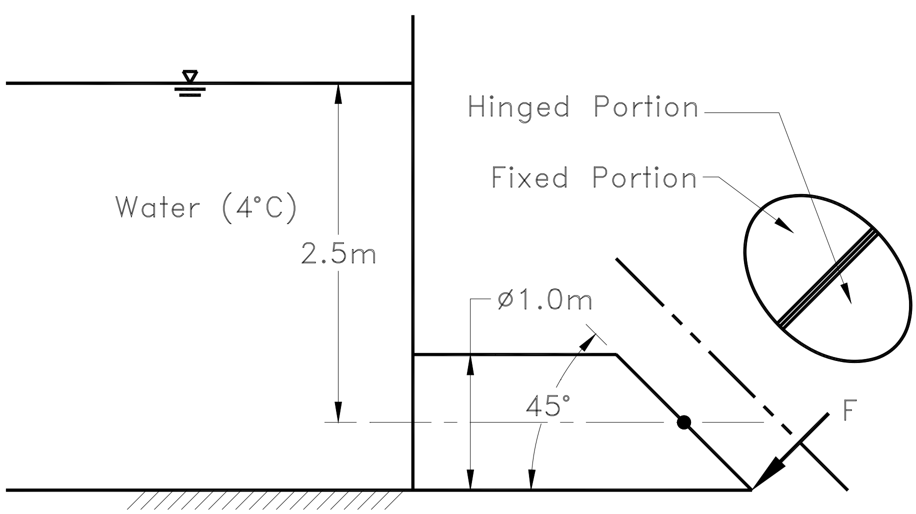
a.) at the water surface.

b.) 0.50 m below the water surface.

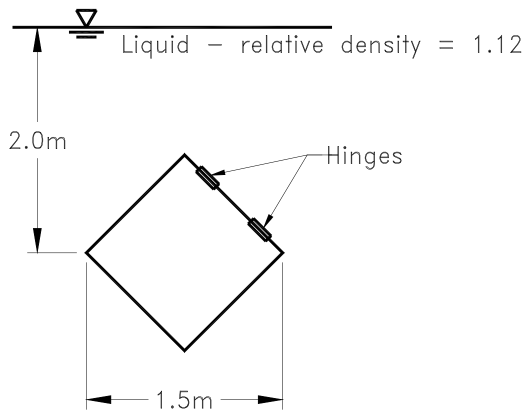
c.) 10.0 m below the water surface.



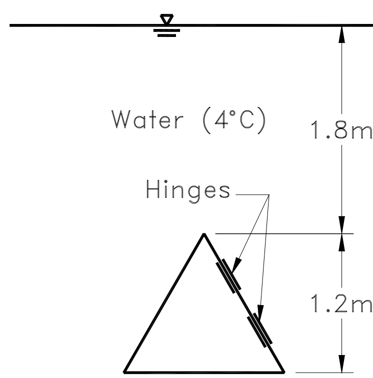
3.8) A 1.0 m diameter pipe leads from a reservoir as shown in the diagram below. The pipe is cut off at 45° and a plate is welded to cover the top half of the pipe, denoted the fixed portion in the diagram. The bottom half of the pipe has a hinged gate. Determine the necessary force F acting perpendicular to the hinged plate at its bottom edge that will keep the hinged gate in the closed position.



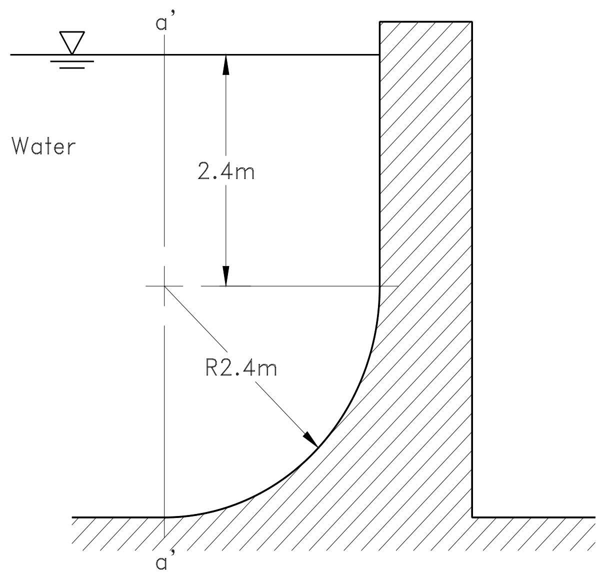
3.9) A square opening in the vertical side of a tank is orientated such that its diagonals are horizontal and vertical. The opening is completely covered by a flat plate hinged along one of the upper sides of the opening. The tank contains a liquid with a relative density 1.12. Calculate the hydrostatic thrust acting on the plate, the position of the center of pressure and the moment of the thrust about the hinged side.



3.10) An equilateral triangular opening in the vertical side of a tank is orientated as shown in the diagram below. The opening is completely covered by a flat plate hinged along one of the upper sides of the opening. Calculate the hydrostatic thrust acting on the plate, the position of the centre of pressure and the moment of the thrust about the hinged side.



3.11) A small dam is shown in the figure below. The dam is 4.0 m long. Determine the resultant hydrostatic thrust acting on the dam, locating its line of action and giving its direction. Show the result on a sketch of the dam. The water behind the dam can be assumed to be 4°C.



3.12) A small dam is shown in the figure below. The dam is 8.0 m long. Determine the resultant hydrostatic thrust acting on the dam, locating its line of action and giving its direction. Show the result on a sketch of the dam. The water behind the dam can be assumed to be 4°C.