Vertex-coloring edge-weightings of graphs

Gerard J. Chang123∗ Changhong Lu4,5† Jiaojiao Wu1‡ Qinglin Yu6§

1Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan
2Institute for Mathematical Sciences, National Taiwan University, Taipei 10617, Taiwan
3National Center for Theoretical Sciences, Taipei, Taiwan
4Department of Mathematics, East China Normal University, Shanghai 200062, P. R. China
5Institute of Theoretical Computing, ECNU, Shanghai 200062, P. R. China
6Department of Mathematics and Statistics, Thompson Rivers University, Kamloops, Canada

Abstract

A k-edge-weighting of a graph G is a mapping w : E(G) → {1, 2, . . . , k}. An edge-weighting w induces a vertex coloring f_w : V(G) → N defined by f_w(v) = ∑_{e ∈ v} w(e). An edge-weighting w is vertex-coloring (respectively, vertex-injective) if f_w(u) ≠ f_w(v) for any edge uv (respectively, every pair of distinct vertices u and v). The current paper studies the parameter µ(G), which is the minimum k for which G has a vertex-coloring k-edge-weighting. Exact values of µ(G) are determined for several classes of graphs, including trees and r-regular bipartite graph with r ≥ 3.

Keywords. Edge-weighting; vertex-coloring; tree; bipartite graph.

1 Introduction

A k-edge-weighting of a graph G is a mapping w : E(G) → {1, 2, . . . , k}. An edge-weighting w induces a vertex coloring f_w : V(G) → N defined by f_w(v) = ∑_{e ∈ v} w(e). An edge-weighting w is vertex-coloring (respectively, vertex-injective) if f_w(u) ≠ f_w(v) for any edge uv (respectively, every pair of distinct vertices u and v). Denote by µ(G) (respectively, µ∗(G)) the minimum k for which G has a vertex-coloring (respectively, vertex-injective) k-edge-weighting. We refer a graph non-trivial if it contains no single edge as a component. Notice that µ(G) ≤ µ∗(G) for every non-trivial graph G.

An edge-weighting is adjacent vertex-distinguishing (respectively, vertex-distinguishing) if for any edge uv (respectively, every pair of distinct vertices u and v), the multi-set of

∗E-mail: gjchang@math.ntu.edu.tw.
†E-mail: chlu@math.ecnu.edu.cn.
‡E-mail: wujj0007@yahoo.com.tw.
§E-mail: yu@tru.ca.
weights appearing on edges incident to $u$ is distinct from the multi-set of weights appearing on the edges incident to $v$. Denote by $\mu_m(G)$ (respectively, $\mu_m^*(G)$) the minimum $k$ for which $G$ has an adjacent vertex-distinguishing (respectively, vertex-distinguishing) $k$-edge-weighting. Notice that $\mu_m(G) \leq \mu_m^*(G)$ for every non-trivial graph $G$. Then, upper bounds for $\mu(G)$ (respectively, $\mu^*(G)$) provide upper bounds for $\mu_m(G)$ (respectively, $\mu_m^*(G)$).

It is clear that a vertex-coloring (respectively, vertex-injective) edge-weighting is adjacent vertex-distinguishing (respectively, vertex-distinguishing), but the converse is not necessarily true. Consequently, $\mu_m(G) \leq \mu(G)$ and $\mu_m^*(G) \leq \mu^*(G)$ for every non-trivial graph $G$.

Adjacent vertex-distinguishing edge-weighting and vertex-distinguishing edge-weighting have been studied by many researchers [4, 6, 5, 7]. Karoński, Luczak and Thomason [10] proved that $\mu_m(G) \leq 213$ for every non-trivial graph and that $\mu_m(G) \leq 30$ for every graph with minimum degree at least $10^{99}$. Addario-Berry et al. [1] improved the results to $\mu_m(G) \leq 4$ for every non-trivial graph and $\mu_m(G) \leq 3$ for every graph of minimum degree at least $1000$.

For vertex-coloring edge-weighting, Karoński, Luczak and Thomason [10] posed the following question:

**Question.** Does $\mu(G) \leq 3$ for every non-trivial graph $G$?

Karoński, Luczak and Thomason [10] showed that if $G$ is a $k$-colorable graph with $k$ odd then $G$ admits a vertex-coloring $k$-edge-weighting. So, for the class of 3-colorable graphs, including bipartite graphs, the answer is affirmative. However, in general, this question is still open. The first constant bound was obtained by Addario-Berry et al. [2], who showed that $\mu(G) \leq 30$ for every non-trivial graph $G$. The bound is improved to $\mu(G) \leq 16$ in [3], to $\mu(G) \leq 13$ in [11], and to $\mu(G) \leq 5$ in [9].

Even we are still far from providing a positive answer to the question, actually $\mu(G) \leq 2$ for many graphs (in fact, experiments suggest (see [10]) that $\mu(G) \leq 2$ for almost all graphs). The current paper is devoted to study graphs with such a property. We determine $\mu(G)$ for some classes of graphs with this property, including trees and $r$-regular bipartite graphs with $r \geq 3$. 

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In the rest of this section, we fix some notation. For \( n \geq 1 \), the \( n \)-path \( P_n \) is the graph with vertex set \( \{v_i : 1 \leq i \leq n\} \) and edge set \( \{v_i v_{i+1} : 1 \leq i \leq n-1\} \). For \( n \geq 3 \), the \( n \)-cycle \( C_n \) is the graph with vertex set \( \{v_i : 1 \leq i \leq n\} \) and edge set \( \{v_i v_{i+1} : 1 \leq i \leq n\} \), where \( v_{n+1} = v_1 \). The complete graph \( K_n \) is the graph with vertex set \( \{v_i : 1 \leq i \leq n\} \) and edge set \( \{v_i v_j : 1 \leq i < j \leq n\} \). The complete bipartite graph \( K_{m,n} \) is the graph with vertex set \( \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\} \) and edge set \( \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \). The neighborhood of a vertex \( v \) is the set \( N(v) = \{u : uv \in E(G)\} \), and the closed neighborhood is \( N[v] = N(v) \cup \{v\} \). The degree of a vertex \( v \) is \( d(v) = |N(v)| \). We use \( \delta(G) \) to denote the minimum degree of a vertex in a graph \( G \).

2 \( \mu(G) \) for some classes of graphs

This section establishes values of \( \mu(G) \) for some classes of graphs, including paths, cycles, complete graphs and complete bipartite graphs.

Fact 1 For every non-trivial graph \( G \), \( \mu(G) = 1 \) if and only if \( G \) has no adjacent vertices with the same degree.

Fact 2 \( \mu(P_3) = 1 \) and \( \mu(P_n) = 2 \) for \( n \geq 4 \).

Proof. This follows from Fact 1 and the fact that the following mapping \( w \) is a vertex-coloring 2-edge-weighting: \( w(v_i v_{i+1}) = 1 \) for \( i \equiv 1, 2 \pmod{4} \) and \( w(v_i v_{i+1}) = 2 \) for \( i \equiv 3, 4 \pmod{4} \).

Proposition 3 \( \mu(C_n) = 2 \) for \( n \equiv 0 \pmod{4} \) and \( \mu(C_n) = 3 \) for \( n \not\equiv 0 \pmod{4} \).

Proof. First, \( \mu(C_n) \geq 2 \) by Fact 1. For the case when \( n \equiv 0 \pmod{4} \), \( \mu(C_n) = 2 \) follows from that the following mapping \( w \) is a vertex-coloring 2-edge-weighting: \( w(v_i v_{i+1}) = 1 \) for \( i \equiv 1, 2 \pmod{4} \) and \( w(v_i v_{i+1}) = 2 \) for \( i \equiv 3, 4 \pmod{4} \).

For the case \( n = 4k+r, 1 \leq r \leq 3 \), \( \mu(C_n) \leq 3 \) follows from that the following mapping \( w \) is a vertex-coloring 3-edge-weighting: \( w(v_i v_{i+1}) = 1 \) for \( i \equiv 1, 2 \pmod{4} \) and \( w(v_i v_{i+1}) = 2 \) for \( i \equiv 3, 4 \pmod{4} \) with the modifications that \( w(v_{4k+1} v_{4k+2}) = w(v_{4k+2} v_{4k+3}) = 3 \) and
$w(v_{4k+3}v_{4k+4}) = 2$. On the other hand, we claim that $\mu(C_n) \neq 2$. Suppose to the contrary that $C_n$ has a vertex-coloring 2-edge-weighting $w$. Then, $f_w(v_{i+1}) \neq f_w(v_{i+2})$ implies $w(v_iv_{i+1}) \neq w(v_{i+2}v_{i+3})$ and so $w(v_iv_{i+1}) = w(v_{i+4}v_{i+5})$, where the indices are taken modulo 4. These in turn imply that $w(v_iv_{i+1}) \neq w(v_{i+4j+2}v_{i+4j+3})$. This is a contradiction since $v_i = v_{i+n} = v_{i+4j+2}$ when $r = 2$ with $j = \frac{n-2}{4}$ and $v_i = v_{i+2n} = v_{i+4j+2}$ when $r = 1, 3$ with $j = \frac{n-1}{2}$.

**Proposition 4** If $n \geq 3$, then $\mu(K_n) = 3$.

**Proof.** We first consider the following mapping $w$: $w(v_iv_j) = 1$ for $i + j \leq n$, $w(v_iv_n) = 3$ for $\left\lfloor \frac{n+2}{2} \right\rfloor \leq i \leq n - 1$, and $w(v_iv_j) = 2$ for all other edges. It is straightforward to check that $f_w(v_i) = n - 1 + i$ for $1 \leq i \leq n - 1$ and $f_w(v_n) = \left\lfloor \frac{5n-5}{2} \right\rfloor$. Hence, $f_w$ is vertex-coloring and so $\mu(K_n) \leq 3$.

On the other hand, we claim that $\mu(K_n) \neq 2$. Suppose to the contrary that $K_n$ has a vertex-coloring 2-edge-weighting $w$. Then, each $f_w(v_i)$ is one of the $n$ possible values in $\{n-1, n, \ldots, 2n-2\}$. So, there is exactly one $v_i$ (resp. $v_j$) with $f_w(v_i) = n-1$ (resp. $f_w(v_j) = 2n-2$). The first equation implies that $w(v_iv_j) = 1$ while the second one implies that $w(v_jv_i) = 2$, a contradiction. Thus, $\mu(K_n) = 3$.

**Proposition 5** $\mu(K_{m,n}) = 1$ when $m \neq n$ and $\mu(K_{m,n}) = 2$ when $m = n \geq 2$.

**Proof.** The former case follows from Fact 1. The latter case follows from that for $m = n \geq 2$ the following mapping $w$ is a vertex-coloring 2-edge-weighting: $w(u_iv_j) = 1$ and $w(u_mv_j) = 2$ for $1 \leq i \leq m - 1$ and $1 \leq j \leq n$.

The *theta graph* $\theta(\ell_1, \ell_2, \ldots, \ell_r)$ is the graph obtained from $r$ disjoint paths of lengths $\ell_1, \ell_2, \ldots, \ell_r$, respectively, by identifying their end-vertices called the *roots* of the graph. Notice that $\theta(\ell_1) = P_{1+\ell_1}$ and $\theta(\ell_1, \ell_2) = C_{\ell_1+\ell_2}$. In the following we only consider the case $r \geq 3$ and assume that $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_r$.

**Proposition 6** Let $G = \theta(\ell_1, \ell_2, \ldots, \ell_r)$ with $r \geq 3$. Then $\mu(G) = 1$ when $\ell_i = 2$ for all $i$; $\mu(G) = 3$ when $\ell_1 = 1$ and $\ell_i \equiv 1 \pmod{4}$ for all $i \neq 1$; and $\mu(G) = 2$ otherwise.
Proof. The first equality follows from Proposition 1 and that any two adjacent vertices have different degrees if and only if all $\ell_i = 2$.

For the case when $\ell_1 = 1$ with all $\ell_i \equiv 1 \pmod{4}$, we claim that $\mu(G) \geq 3$. Suppose, to the contrary that the graph admits a vertex-coloring 2-edge-weighting $w$. Then, in each path the $k$th edge must have the different weight from the $(k+2)$th edge, and has the same weight with the $(k+4)$th edge. Consequently, the first edge has the same weight with the last edge in each path of the theta graph. Then, $f_w(u) = f_w(v)$ for the two roots $u$ and $v$, however, this is impossible as they are adjacent. On the other hand, the following mapping $w$ is a vertex-coloring 3-edge-weighting: for each path of the theta graph, assign the weights 1, 1, 2, 2 periodically except the last edge assigned with 3.

For the remaining case, we may construct a vertex-coloring 2-edge-weighting as follows. Notice that for a periodical weight assignment $\ldots, 1, 1, 2, 2, \ldots$ of a path with first edge $e$ and last edge $e'$, we may properly choose the starting weight such that $w(e) = w(e') = 2$ except for the case when $\ell_i \equiv 3 \pmod{4}$ (one of $w(e)$ and $w(e')$ is 1 and the other is 2). We then may properly arrange the weights on edges to make a vertex-coloring 2-edge-weighting even when $\ell_1 = 1$.

\section{$\mu(G)$ for bipartite graphs}

In this section, we consider $\mu(G)$ for a bipartite graph $G$. We use $G = (A, B, E)$ to denote a bipartite graph with vertex bipartition $(A, B)$, and edge set $E$.

\textbf{Theorem 7} Every non-trivial connected bipartite graph $G = (A, B, E)$ with $|A|$ even admits a vertex-coloring 2-edge-weighting $w$ such that $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B$. Consequently, $\mu(G) \leq 2$.

\textbf{Proof.} Assume that $A = \{a_1, a_2, \ldots, a_{2r}\}$. Let $P_i$ be a path from $a_i$ to $a_{r+i}$ for $1 \leq i \leq r$. For each edge $e$, denote $k(e)$ the number of such paths containing $e$; and for each vertex $u$, denote $m(u)$ the sum of $k(e)$ of all edges $e$ incident to $u$. Then $m(u)$ is odd for $u \in A$ and $m(v)$ is even for $v \in B$. Now, let $w(e) = 1$ for any edge $e$ with $k(e)$ odd and $w(e) = 2$ for any edge $e$ with $k(e)$ even. Since $w(e)$ has the same parity as $k(e)$ for each edge $e$, the color
\( f_w(u) \) of a vertex \( u \) satisfies \( f_w(u) \equiv m(u) \pmod{2} \) for \( u \in A \cup B \). Consequently, \( f_w(u) \) is odd for \( u \in A \) and \( f_w(v) \) is even for \( v \in B \). Hence, \( w \) is a vertex-coloring 2-edge-weighting of \( G \).

**Theorem 8** \( \mu(G) \leq 2 \) for every non-trivial connected bipartite graph \( G = (A, B, E) \) with \( \delta(G) = 1 \).

**Proof.** By Theorem 7, we may assume that both of \( |A| \) and \( |B| \) are odd. Without loss of generality, assume that \( d(x) = 1 \) for some vertex \( x \) in \( A \), and that \( x \) is adjacent to a vertex \( y \) in \( B \). Then \( G - x = (A \setminus \{x\}, B, E \setminus \{xy\}) \) is a non-trivial connected bipartite graph with \( |A - \{x\}| \) even. By Theorem 7, \( G - x \) has a 2-edge-weighting \( w' \) so that \( f_{w'}(u) \) is odd for \( u \in A \setminus \{x\} \) and \( f_{w'}(v) \) is even for \( v \in B \). Now, extend \( w' \) to \( w \) for \( G \) by assigning \( w(xy) = 2 \). This gives a vertex-coloring 2-edge-weighting with \( f_w(x) = 2, f_w(u) \) odd for \( u \in A \setminus \{x\}, f_w(v) \) even for \( v \in B \) and \( f_w(y) > 2 \).

**Corollary 9** If \( T \) is a tree of at least three vertices, then \( \mu(T) \leq 2 \).

**Theorem 10** \( \mu(G) \leq 2 \) for every non-trivial connected bipartite graph \( G = (A, B, E) \) if \( \lfloor d(u)/2 \rfloor + 1 \neq d(v) \) for any edge \( uv \in E(G) \).

**Proof.** By Theorem 7, we may assume that both of \( |A| \) and \( |B| \) are odd. We need a claim first.

*Claim.* There exists a vertex \( x \), say \( x \in B \), such that the vertices of \( G - N[x] \) in \( A \) are all in a same component of \( G - N[x] \).

Choose a vertex \( x \) such that the size of a maximum component of \( G - N[x] \) becomes as large as possible. Without loss of generality, we assume that \( x \in B \). Suppose that besides a maximum component \( G_1 = (A_1, B_1, E_1) \) the graph \( G - N[x] \) has another component \( G_2 = (A_2, B_2, E_2) \), where \( A_1 \) and \( A_2 \) are nonempty subsets of \( A \). Choose \( x' \in A_2 \). Since \( G \) is connected, \( N(x) \) has a vertex adjacent to a vertex in \( B_1 \). Then, \( G_1 \) together with \( N[x] \) are in a same component of \( G - N[x'] \), and then the size of a maximum component of \( G - N[x'] \) is larger than that of \( x \), a contradiction to the choice of \( x \).
From the claim, we see that $G - N(x)$ has a component $G_1 = (A_1, B_1, E_1)$ with $A_1 = A \setminus N(x)$ and all other components are isolated vertices in $B$. Now we consider two cases.

**Case 1.** $d(x)$ is odd. In this case, $|A_1|$ is even. According to Theorem 7, $G_1$ has a 2-edge-weighting $w'$ such that $f_{w'}(u)$ is odd for $u \in A_1$ and $f_{w'}(v)$ is even for $v \in B_1$. We then extend $w'$ to $w$ for $G$ by assigning the edges incident to $x$ with weight 1 and the remaining edges with weight 2. Then, $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B \setminus \{x\}$. Notice that $f_w(x) = d(x)$ and $f_w(u) = 2d(u) - 1$ for all $u \in N(x)$. These imply $f_w(x) \neq f_w(u)$ by hypothesis. Therefore, $w$ is a vertex-coloring 2-edge-weighting of $G$.

**Case 2.** $d(x)$ is even. In this case, $|A_1|$ is odd. Notice that there is a vertex $u^* \in N(x)$ adjacent to some vertex $v^* \in B_1$. Let $G'$ be the graph obtained from $G_1$ by adding the vertex $u^*$ and the edge $u^*v^*$. According to Theorem 7, $G'$ has a 2-edge-weighting $w'$ so that $f_{w'}(u)$ is odd for $u \in A_1 \cup \{u^*\}$ and $f_{w'}(v)$ is even for $v \in B_1$. We may extend $w'$ to $w$ for $G$ by assigning the edges incident to $x$, except $xu^*$, with weight 1 and the remaining edges with weight 2. Then, $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B$ except $x$. Notice that $f_w(x) = 2\lfloor d(x)/2 \rfloor + 1$ for all $u \in N(x) - u^*$. Therefore, $w$ is a vertex-coloring 2-edge-weighting of $G$. 

Consequently, we have the following result which is in fact our first thought.

**Corollary 11** $\mu(G) = 2$ for every $r$-regular bipartite graph $G$ with $r \geq 3$.

Notice that the theta graph $G = \theta(\ell_1, \ell_2, \ldots, \ell_r)$ with $\ell_1 = 1$ and all $\ell_i \equiv 1 \pmod{4}$ is a bipartite graph with $\mu(G) = 3$.

We conclude the paper by posing the following problem.

**Problem.** Characterize bipartite graphs with vertex-coloring 2-edge-weighting.

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References


