Exact solutions in front propagation problems with superdiffusion

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Abstract

Front propagation in a number of reaction-superdiffusion problems is studied. Specifically, traveling wave propagation, domain wall pinning, and systems of waves governed by a bistable single equation as well as FitzHugh-Nagumo equations are considered. The reaction terms in the equations are taken in the form of piecewise linear functions, which allows for exact solutions to be obtained. The effect of superdiffusion on front propagation is discussed.

Key words: reaction-diffusion problem, superdiffusion, front, FitzHugh-Nagumo equations, exact solution
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1. Introduction

Traveling wave solutions of reaction-diffusion problems attract much attention of researchers due to both numerous applications and mathematical challenges which are mainly faced when systems of equations are considered rather than single equations. Studies of traveling wave solutions began with the Fisher [1] – Kolmogorov-Petrovsky-Piskunov (KPP) [2] equation on the

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one hand and Zeldovich and Frank-Kamenetskii work on combustion problems [3] on the other. Since then, thousands of papers on traveling wave solutions have been published (see the monographs [4–18] and the references therein).

A characteristic feature of most of the reaction-diffusion systems that have been studied to date is that diffusion is normal, i.e., at the molecular level it is the result of independent random jumps, e.g., nearest neighbor jumps, at regularly spaced time increments. In fact, the molecules can wait between successive jumps and can also execute not just nearest neighbor jumps, but rather long jumps. However, both the waiting time distribution and jump length distribution must have finite moments for normal diffusion to occur. In some cases, however, these conditions are not met, in that the molecules may undergo *anomalous diffusion* [19–24]. Unlike normal diffusion, which is characterized by the dependence $< (\Delta \mathbf{r})^2 > \sim t$ of the mean square displacement of a randomly walking particle on time, anomalous diffusion is characterized by the more general dependence

$$< (\Delta \mathbf{r})^2 >= 2dK_\alpha t^\alpha,$$

where d is the (embedding) spatial dimension, K_α is a generalized diffusion constant, and the exponent α is not necessarily an integer. For $\alpha = 1$ anomalous diffusion reduces to normal diffusion, with K_1 being the ordinary diffusion coefficient. For $\alpha < 1$ ($\alpha > 1$), the diffusion process is slower (faster) than normal diffusion and is called "subdiffusive" ("superdiffusive"). Both types of anomalous diffusion processes have been recognized to play important roles in various physical, chemical, biological and geological processes. For example, subdiffusion, which corresponds to molecules waiting for long times before jumping, i.e., with a waiting time distribution having infinite moments, often occurs in gels (especially bio-gels [25, 26]), porous media [27] and polymers [28], while superdiffusion occurs in systems where there are long jumps of particles, i.e., with a jump length distribution having infinite moments. It is typical of some processes in plasmas and turbulence [29, 30], surface diffusion [31, 32], charge carrier transfer in semiconductors [33], as well as in geophysical and geological processes, including the dispersion of nuclear waste in soil [34] (see also [19–24] for reviews and numerous other examples). A special case of superdiffusive process corresponds to Lévy flights [24].

Although many aspects of anomalous diffusion have been extensively studied (see [24] for a recent review), the propagation of reaction-diffusion

fronts governed by anomalous diffusion was a subject of only a very limited number of works. A problem with nonlinear kinetics similar to that in the KPP problem [2] was investigated both for the subdiffusive case [35, 36] and the superdiffusive case [37–39]. It was shown, in particular, that superdiffusion leads to significant acceleration of the front speed. Propagation failure in the subdiffusive case was discussed in [40]. Superdiffusive front dynamics in the Allen-Cahn equation was analyzed in [41]. Another study [42] of the bistable case focused on directional anomalous diffusion.

In this paper we consider several front propagation problems in the bistable case. In all these problems the reaction term has a sufficiently simple piecewise linear form, so that exact solutions of the problems can be found.

2. Mathematical model

We consider the following reaction-superdiffusion equation

$$\partial_t w = D^\gamma_{|x|} w + f(w). \tag{1}$$

The operator $D^\gamma_{|x|}$ represents the superdiffusion term and is defined by its action in the Fourier space as

$$\mathcal{F}_{x \to q}\left\{ D^\gamma_{|x|} w(x,t) \right\} = -|q|^\gamma \mathcal{F}_{x \to q}\left\{ w(x,t) \right\}, \quad 1 \leq \gamma \leq 2.$$

The nonlinear term f is a piecewise linear function

$$f(w) = -k[w - H(w - a)] = \begin{cases} -kw, & 0 \leq w < a \\ -k(w-1), & a < w \leq 1 \end{cases}$$

(see Figure 1). Here $k > 0$ and $0 < a < 1$ are two source term parameters, and H is the Heaviside function. The traveling wave $w(x - ct)$ is then governed by the equation

$$D^\gamma_{|x|} w + c\frac{dw}{dx} + f(w) = 0, \quad -\infty < x < \infty, \tag{2a}$$

with the boundary conditions at infinities

$$w(-\infty) = 0, \quad w(\infty) = 1. \tag{2b}$$

Due to the translational invariance of the solution, we can assume that
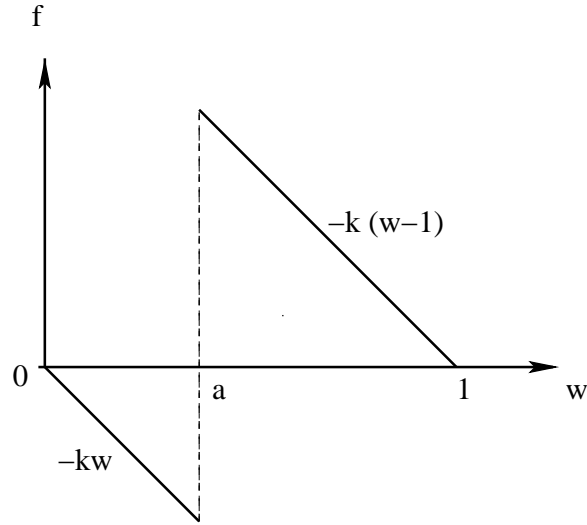
$$w(0) = a. \tag{3}$$

Figure 1: The source term in the equation (1)

We remark that we are seeking a monotonically increasing solution $w(x)$ (Figure 2). Then $w(x) < a$ for $x < 0$, and $w(x) > a$ for $x > 0$. Thus

$$H(w(x) - a) = H(x),$$

and the source term can be written as

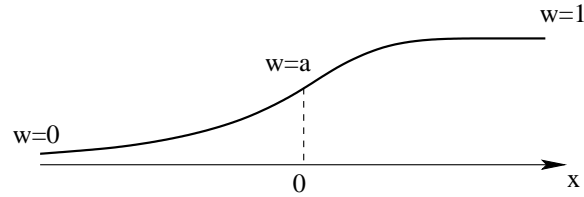$$f(w) = -k[w - H(x)]. \tag{4}$$

Figure 2: The sketch of the traveling wave solution

We next apply the Fourier transform to equation (2a), which, as follows from (4), is a linear nonhomogeneous equation, to obtain

$$(-|q|^\gamma + ciq - k)\widetilde{w} + k\left(\pi\delta(q) - \frac{i}{q}\right) = 0. \tag{5}$$

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Here the Fourier transform

$$\widetilde{w}(q) = \mathcal{F}_{x \to q}\{w(x)\} = \int_{-\infty}^{\infty} w(x)e^{-iqx}\,dx,$$

is understood in the sense of distributions, in particular we have used

$$\mathcal{F}_{x \to q}\{H(x)\} = \pi\delta(q) - \frac{i}{q},$$

where δ is the Dirac δ-function. Solving (5) for $\widetilde{w}$ and inverting the Fourier transform yields

$$
\begin{aligned}
w(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k(\pi\delta(q) - i/q)}{|q|^\gamma - ciq + k}e^{iqx}\,dq \\
&= \frac{1}{2} - \frac{ki}{2\pi}\,V.P.\int_{-\infty}^{\infty}\frac{1}{q(|q|^\gamma - ciq + k)}e^{iqx}\,dq \\
&= \frac{1}{2} - \frac{ki}{2\pi}\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon}\frac{e^{iqx}\,dq}{q(|q|^\gamma - ciq + k)}\right. \\
&\quad + \left.\int_{\epsilon}^{\infty}\frac{e^{iqx}\,dq}{q(|q|^\gamma - ciq + k)}\right] \\
&= \frac{1}{2} - \frac{ki}{2\pi}\lim_{\epsilon\to 0}\left[-\int_{\epsilon}^{\infty}\frac{e^{-iqx}\,dq}{q(q^\gamma + ciq + k)}\right. \\
&\quad + \left.\int_{\epsilon}^{\infty}\frac{e^{iqx}\,dq}{q(q^\gamma - ciq + k)}\right] \\
&= \frac{1}{2} + \frac{k}{\pi}\,\mathrm{Im}\int_{0}^{\infty}\frac{(q^\gamma + ciq + k)e^{iqx}}{q[(q^\gamma + k)^2 + c^2q^2]}\,dq \\
&= \frac{1}{2} + \frac{k}{\pi}\int_{0}^{\infty}\frac{(q^\gamma + k)\sin(qx) + cq\cos(qx)}{q[(q^\gamma + k)^2 + c^2q^2]}\,dq.
\end{aligned}
$$

In this calculation, $V.P.$ denotes the Cauchy principal value of the integral. Making the substitution $q = sk^{1/\gamma}$ in the last integral, we finally obtain

$$w(x) = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty}\frac{(s^\gamma + 1)\sin(s\xi) + \alpha s\cos(s\xi)}{s[(s^\gamma + 1)^2 + \alpha^2 s^2]}\,ds, \tag{6}$$

where

$$\alpha = ck^{1/\gamma - 1}, \quad \xi = xk^{1/\gamma} \tag{7}$$

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are the scaled propagation velocity and spatial variable.

The propagation velocity is determined from the condition (3), which implies

$$a - \frac{1}{2} = \frac{\alpha}{\pi} \int_0^\infty \frac{1}{(s^\gamma + 1)^2 + \alpha^2 s^2} \, ds \equiv F(\alpha; \gamma), \qquad (8)$$

and can be treated as the inverse dependence of the scaled propagation velocity α on the source parameter a. Thus, properties of the function $F(\alpha; \gamma)$ can be translated into the properties of the propagation velocity as a function of the parameters. The function $F(\alpha; \gamma)$ varies from $-1/2$ as $\alpha \to -\infty$ to $1/2$ as $\alpha \to \infty$ and it is an odd, monotonically increasing function of α for any $1 \le \gamma \le 2$ (see Figure 3). It means that there is a unique traveling wave solution of the problem (2) for any $0 < \alpha < 1$, $k > 0$ and $1 \le \gamma \le 2$.
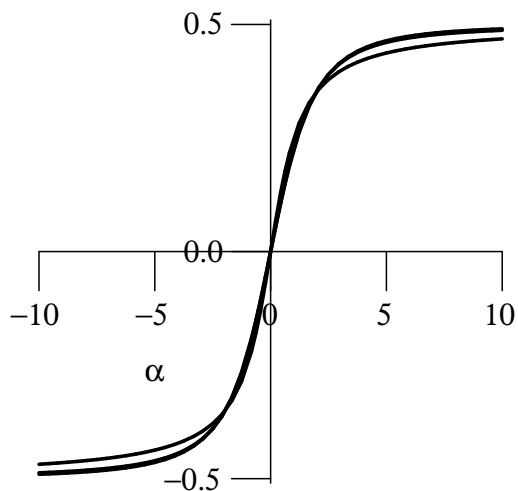

Figure 3: The graph of the function $F(\alpha; \gamma)$ for $\gamma = 2, 1.5, 1$. The graphs of $F(\alpha; 2)$ and $F(\alpha; 1.5)$ are almost indistinguishable, i.e., the propagation speed does not noticeably depend on γ unless γ is sufficiently close to one. For large α, $F(\alpha; 2) > F(\alpha; 1)$.

As a function of γ, F is monotonically increasing for α large (it is sufficient to take $\alpha > 3/2$) and is monotonically decreasing for $\alpha > 0$ close to zero (it is sufficient to take $0 < \alpha < 1/2$). For $\gamma = 1$ and $\gamma = 2$ the function can be

expressed in terms of elementary functions as

$$F(\alpha; 1) = \frac{1}{\pi} \arctan \alpha, \quad F(\alpha; 2) = \frac{\alpha}{2\sqrt{4 + \alpha^2}}.$$

Traveling wave propagation represents the process of displacement of one steady state by another. We will refer to the establishing steady state as the dominant. If $c > 0$, then the wave goes to the right so that the state $w = 0$ displaces the state $w = 1$ and, therefore, is the dominant one. If $c < 0$, then the wave goes to the left and $w = 1$ is the dominant state. It is easy to see from the graphs of F and equation (8) that which state dominates depends on whether a is greater or less than one half, or, equivalently, on the sign of the integral of the source term

$$\int_0^1 f(w) \, dw.$$

A positive (negative) integral implies that $c < 0$ ($c > 0$) and $w = 1$ ($w = 0$) dominates. It can be shown that similar to what is known for normal diffusion, the direction of propagation of the wave is determined by the sign of the integral of the source term not only for the specific piecewise linear function that we consider, but for any $f(w)$ provided that the wave exists. Indeed, multiplying equation (2a), where $f(w)$ is a general source term, by dw/dx and integrating from $-\infty$ to ∞ in x yields

$$\int_{-\infty}^{\infty} D_{|x|}^{\gamma} w \, \frac{dw}{dx} \, dx + c \int_{-\infty}^{\infty} \left(\frac{dw}{dx} \right)^2 dx + \int_0^1 f(w) \, dw = 0.$$

The first integral is equal to zero. Indeed,

$$\int_{-\infty}^{\infty} D_{|x|}^{\gamma} w \, \frac{dw}{dx} \, dx =$$

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dw}{dx} \left(\int_{-\infty}^{\infty} |q|^{\gamma} \widetilde{w}(q) e^{iqx} \, dq \right) dx =$$

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} |q|^{\gamma} \widetilde{w}(q) \left(\int_{-\infty}^{\infty} \frac{dw}{dx} e^{iqx} \, dx \right) dq =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} iq |q|^{\gamma} \widetilde{w}(q) \overline{\widetilde{w}(q)} \, dq = 0$$

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because the integrand is an odd function of q. Thus,

$$c = -\int_0^1 f(w)\,dw\left[\int_{-\infty}^\infty \left(\frac{dw}{dx}\right)^2 dx\right]^{-1},$$

i.e., the sign of c is the opposite of the sign of the integral of the source function.

Figure 3 also demonstrates that unless $|\alpha|$ is large or, equivalently, a is close to zero or to one, the graphs of F for different values of γ are very close to each other, so that the propagation velocity in this case is not sensitive to the type of the diffusion process. The proximity of a to zero or to one can be interpreted as a significant dominance of one of the states over the other. Thus, if neither of the states is significantly dominant, the propagation velocity is almost independent of the diffusion exponent.

When one of the states significantly dominates the other (a is close to zero or to one), the situation is quite different. Not only the propagation velocities are numerically significantly different for different γ, but they also obey different power laws. This statement can be quantified by determining the asymptotics of F for large α. We have

$$\begin{aligned}
F(\alpha,\gamma) &= \frac{1}{\pi}\int_0^\infty \frac{dt}{(1+t^\gamma/\alpha^\gamma)^2 + t^2}\\
&\sim \frac{1}{\pi}\int_0^\infty \frac{dt}{1+t^2+2t^\gamma/\alpha^\gamma}\\
&\sim \frac{1}{\pi}\int_0^\infty \frac{1-2t^\gamma/\alpha^\gamma/(1+t^2)}{1+t^2}\,dt\\
&= \frac{1}{2} - \frac{2}{\pi}\frac{1}{\alpha^\gamma}\int_0^\infty \frac{t^\gamma}{(1+t^2)^2}\,dt\\
&= \frac{1}{2} + \frac{1}{2}\frac{1}{\alpha^\gamma}\frac{\gamma-1}{\cos(\pi\gamma/2)}.
\end{aligned}$$

Thus, for a close to 1, which corresponds to the case of large α, the propagation velocity can be approximately determined from

$$a - 1 \approx \frac{1}{2}\frac{1}{\alpha^\gamma}\frac{\gamma-1}{\cos(\pi\gamma/2)}$$

yielding

$$\alpha \approx \left(\frac{\gamma-1}{2(1-a)\sin(\pi(\gamma-1)/2)}\right)^{1/\gamma}.$$

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In particular, for $\gamma = 1$

$$\alpha \approx \frac{1}{\pi(1-a)}$$

and for $\gamma = 2$

$$\alpha \approx \frac{1}{\sqrt{2(1-a)}}.$$

It is instructive to study the behavior of the solution $w(x)$ for large $|x|$. This behavior is important, in particular, as it affects the interaction of two domain walls (cf. [41]). Using (6) we can write $w(x)$ as

$$w(x) = \frac{1}{2} + \frac{1}{\pi}I_1 - \frac{\alpha^2}{\pi}I_2 + \frac{\alpha}{\pi}I_3,$$

where

$$I_1 = \int_0^\infty \frac{\sin(s\xi)}{s(s^\gamma + 1)}\,ds,$$

$$I_2 = \int_0^\infty \frac{s\sin(s\xi)}{(s^\gamma + 1)[(s^\gamma + 1)^2 + \alpha^2 s^2]}\,ds,$$

$$I_3 = \int_0^\infty \frac{\cos(s\xi)}{(s^\gamma + 1)^2 + \alpha^2 s^2}\,ds,$$

and we study each of the integrals separately. We have

$$\begin{aligned}
I_1 &= \int_0^\infty \frac{\sin(s\xi)}{s}\, ds - \int_0^\infty \frac{s^{\gamma-1}\sin(s\xi)}{s^\gamma+1}\, ds \\
&= \frac{\pi}{2}H(\xi) - \frac{1}{\xi}\int_0^\infty \frac{d}{ds}\left(\frac{s^{\gamma-1}}{s^\gamma+1}\right)\cos(s\xi)\, ds \\
&= \frac{\pi}{2}H(\xi) - \frac{\gamma-1}{\xi}\int_0^\infty \frac{s^{\gamma-2}}{s^\gamma+1}\cos(s\xi)\, ds \\
&\quad + \frac{\gamma}{\xi}\int_0^\infty \frac{s^{2\gamma-2}}{(s^\gamma+1)^2}\cos(s\xi)\, ds \\
&= \frac{\pi}{2}H(\xi) - \frac{\gamma-1}{\xi}\int_0^\infty s^{\gamma-2}\cos(s\xi)\, ds \\
&\quad + \frac{1}{\xi}\int_0^\infty \frac{(\gamma-1)s^{3\gamma-2}+(2\gamma-1)s^{2\gamma-2}}{(s^\gamma+1)^2}\cos(s\xi)\, ds \\
&= \frac{\pi}{2}H(\xi) - \frac{\mathrm{sgn}(\xi)}{|\xi|^\gamma}\sin\left(\frac{1}{2}\pi\gamma\right)\Gamma(\gamma) \\
&\quad + \frac{1}{\xi^2}\int_0^\infty \left(\frac{(\gamma-1)s^{3\gamma-2}+(2\gamma-1)s^{2\gamma-2}}{(s^\gamma+1)^2}\right)'\sin(s\xi)\, ds.
\end{aligned}$$

The function that multiplies $\sin(s\xi)$ in the last integral is absolutely integrable, so that the integral, according to the Riemann-Lebesgue lemma goes to zero as $\xi \to \infty$. Thus, as $\xi \to \infty$

$$I_1 \sim \frac{\pi}{2}H(\xi) - \frac{\mathrm{sgn}(\xi)}{|\xi|^\gamma}\sin\left(\frac{1}{2}\pi\gamma\right)\Gamma(\gamma).$$

Next, we show that the integrals I_2 and I_3 do not contribute to the leading order asymptotics. Indeed, integrating I_2 by parts twice we obtain

$$I_2 = \frac{1}{\xi^2}\int_0^\infty \left(\frac{s}{(s^\gamma+1)[(s^\gamma+1)^2+\alpha^2 s^2]}\right)''\sin(s\xi)\, ds.$$

Once again, the function that multiplies $\sin(s\xi)$ in the integral is absolutely integrable, so that the integral goes to zero as $\xi \to \infty$ and

$$I_2 = o\left(\frac{1}{\xi^2}\right).$$

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In a similar way, integrating by parts twice in I_3, we obtain

$$I_3 = \frac{1}{\xi^2} \int_0^\infty \left(\frac{1}{(s^\gamma + 1)^2 + \alpha^2 s^2} \right)'' \cos(s\xi) \, ds,$$

and the function that multiplies $\cos(s\xi)$ is absolutely integrable, so that

$$I_3 = o\left(\frac{1}{\xi^2} \right).$$

We finally obtain for $1 \leq \gamma < 2$

$$w(x) \sim \frac{1}{\pi |\xi|^\gamma} \sin\left(\frac{1}{2}\pi\gamma \right) \Gamma(\gamma), \quad x \to -\infty,$$

$$w(x) \sim 1 - \frac{1}{\pi \xi^\gamma} \sin\left(\frac{1}{2}\pi\gamma \right) \Gamma(\gamma), \quad x \to \infty,$$

with ξ given by (7). This power law behavior is very different from the exponential behavior of the solution in the case of normal diffusion. We remark that $\gamma = 2$ is a singular value in our calculations in the sense that the coefficients of all the power terms of the above expansions vanish.

We finally remark that a more general formulation of the traveling wave problem

$$D_{|x|}^\gamma w + c\frac{dw}{dx} + f(w) = 0, \quad -\infty < x < \infty,$$

$$w(-\infty) = w_1, \quad w(\infty) = w_2$$

with

$$f(w) = -k[w - w_1 + (w_1 - w_2)H(w - a)]$$

can be treated in the same way as (2), yielding the equation for the propagation velocity

$$\frac{a - \frac{1}{2}(w_1 + w_2)}{w_2 - w_1} = F(\alpha, \gamma). \tag{9}$$

where α is the scaled propagation velocity (7).

3. Domain wall pinning

We now discuss a front propagation problem, in which an additional non-homogeneous source term of strength A localized at $x = 0$ is present in the system. As the front propagates from plus or minus infinity, it encounters the obstacle and can either pass it, upon undergoing some transformation, or get stuck, thus resulting in the existence of a stationary solution of the problem.

This problem may been given a physical interpretation as domain wall "pinning". This term originates from the magnetism theory, where a domain wall is an interface between two magnetic domains that can propagate, thus causing the displacement of one of the domains by the other. Non-magnetic inclusions in the medium can stop domain wall propagation, which is referred to as domain wall pinning. For a review of defect pinning phenomena in normal diffusion-reaction systems, see [43].

We study the existence of the stationary solutions, which we interpret as the inability of the front to propagate, i.e., the front pinning. We consider the problem

$$D_{|x|}^{\gamma} w - k[w - H(w - a)] + A\delta(x) = 0, \quad -\infty < x < \infty, \qquad (10)$$

where $\delta(x)$ is the Dirac δ-function, the strength A of which can be either positive or negative. The solution w must satisfy the conditions at infinities

$$w(-\infty) = 0, \quad w(\infty) = 1. \qquad (11)$$

Assuming

$$w(x_0) = a \qquad (12)$$

where x_0 is unknown, and taking into account that $w(x)$ is a monotonically increasing function, we see that

$$H(w - a) = H(x - x_0),$$

so that equation (10) can be reformulated as

$$D_{|x|}^{\gamma} w - k[w - H(x - x_0)] + A\delta(x) = 0, \quad -\infty < x < \infty. \qquad (13)$$

Applying the Fourier transform to (13), we obtain

$$(-|q|^{\gamma} - k)\widetilde{w} + k\left(\pi\delta(q) - \frac{i}{q}\right)e^{-iqx_0} + A = 0.$$

Thus,

$$w(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{k(\pi\delta(q) - i/q)\exp(-iqx_0) + A}{|q|^\gamma + k} e^{iqx} \, dq$$

$$= \frac{1}{2} - \frac{ki}{2\pi} V.P. \int_{-\infty}^{\infty} \frac{1}{q(|q|^\gamma + k)} e^{iq(x-x_0)} \, dq$$

$$+ \frac{A}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|q|^\gamma + k} e^{iqx} \, dq$$

$$= \frac{1}{2} + \frac{k}{\pi} \int_0^{\infty} \frac{\sin q(x - x_0)}{q(q^\gamma + k)} \, dq + \frac{A}{\pi} \int_0^{\infty} \frac{\cos(qx)}{q^\gamma + k} \, dq.$$

Evaluating the above result at $x = x_0$, we obtain an equation for x_0

$$a - \frac{1}{2} = \frac{A}{\pi} \int_0^{\infty} \frac{\cos(qx_0)}{q^\gamma + k} \, dq, \tag{14}$$

which can be written in the form

$$a - \frac{1}{2} = \frac{A}{k^{1-1/\gamma}} G(x_1; \gamma) \tag{15}$$

with

$$G(x; \gamma) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos(sx)}{s^\gamma + 1} \, ds, \quad x_1 = x_0 k^{1/\gamma}. \tag{16}$$

Existence of solution of the equation manifests domain pinning, while the absence of solution means that the front can propagate. For $\gamma = 2$, eq. (14) simplifies to

$$a - \frac{1}{2} = \frac{A}{2\sqrt{k}} e^{-|x_0|\sqrt{k}}.$$

For a general γ, the function $G(x; \gamma)$ is also an even function that monotonically decreases for $x > 0$. It is shown in Figure 4 for three values of γ.

The maximum value of G which occurs at $x = 0$ is

$$G(0; \gamma) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{s^\gamma + 1} \, ds = \frac{1}{\gamma \sin(\pi/\gamma)}.$$

Thus, we can draw the following conclusions about the existence of the stationary solution of the problem. If $a > 1/2$, so that the speed of propagation
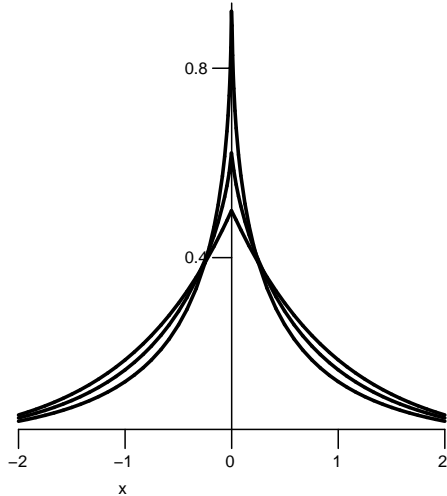
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Figure 4: The graph of the function $G(x; \gamma)$ (see (16) for $\gamma = 2, 5/3, 4/3$. The smaller values of γ correspond to larger values of G at $x = 0$.

without the nonhomogeneity $(A = 0)$ would be positive, the solution exists only if $A > 0$ and moreover, A must be sufficiently large,

$$A > \gamma k^{1-1/\gamma} \left(a - \frac{1}{2}\right) \sin \frac{\pi}{\gamma} \equiv A_{cr}(\gamma)$$

for a solution to exist. This is consistent with our understanding that without the nonhomogeneity for $a > 1/2$ the smaller stationary state $w = 0$ displaces the larger stationary state $w = 1$, because the integral value of the source term is negative. Adding the nonhomogeneous term with a negative A makes the source term even more negative and therefore enhances wave propagation rather than acts to stop the front. If $A > 0$ but not large enough, the nonhomogeneity does act to prevent front propagation but is not sufficiently strong.

If $a < 1/2$, then the situation is similar in the sense that the stationary solution exists only for a sufficiently strong nonhomogeneity of the right sign, namely, for $A < A_{cr}$ (note that $A_{cr} < 0$ in this case). The above qualitative explanation of the existence result that addresses the sign of the source term works here as well.

We remark that $A_{cr}(\gamma)$ monotonically approaches zero as γ decreases from $\gamma = 2$ to $\gamma = 1$. Thus, pinning occurs more readily in the superdiffu-

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sive medium than in the case of normal diffusion. In fact, pinning becomes inevitable in the limit $\gamma \to 1$ since $A_{cr}(1) = 0$.

Finally, we observe that if a stationary solution exists, it is non-unique, because $G(x, \gamma)$ is an even function of x, so that if x_0 is a solution of (15), then $-x_0$ is also a solution. One of these solutions must be unstable. Though we have not performed a linear stability analysis, it is reasonable to expect that the solution, in which the wave already passed the nonhomogeneity (i.e., $x_0 > 0$ for $a > 1/2$ and $x_0 < 0$ for $a < 1/2$), is unstable and can evolve into a front propagating solution.

4. Systems of waves

We now discuss a model equation with a more complex nonlinearity that has three intermediate zeros instead of one. The source term is shown in Figure 5.
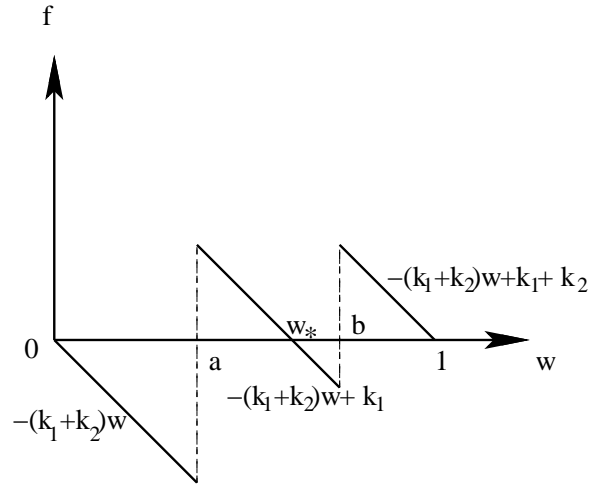

Figure 5: The source term in the equation (17)

According to the results in Section 2, there may be a traveling wave that connects the steady state $w = 0$ and $w = w_* \equiv k_1/(k_1 + k_2)$ (the $[0, w_*]$-wave) as well as a wave that connects $w = w_*$ with $w = 1$ (the $[w_*, 1]$-wave). We are interested in the existence of a $[0, 1]$-wave, i.e., the wave that connects $w = 0$ and $w = 1$ (Figure 6).

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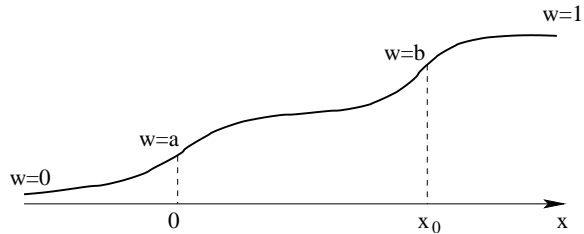
Figure 6: The sketch of the traveling wave solution

Known results [44] state that in the case of normal diffusion a $[0, 1]$-wave exists if the propagation velocity of the $[0, w_*]$-wave exceeds the velocity of the $[w_*, 1]$-wave. The dynamics of the solution of the time-dependent problem is as follows. Suppose the initial condition is composed of two parts that are separated by an approximately $w = w_*$ plateau: one is a profile resembling the $[0, w_*]$-wave and the other $[w_*, 1]$-wave. Then if the velocity of the former is larger than that of the latter, the $[0, w_*]$-wave will catch up to the $[w_*, 1]$-wave and a single wave will form. If the velocity of the $[0, w_*]$-wave is smaller than the velocity of the $[w_*, 1]$-wave, the waves will propagate independently, the distance between the fronts will increase and no $[0, 1]$-wave exists in this case. We want to check if the same result is valid for the superdiffusion problem with the piecewise linear source term. Thus, we consider the problem

$$D_{|x|}^{\gamma} w + cw' - (k_1 + k_2)w + k_1 H(w - a) + k_2 H(w - b) = 0, \qquad (17)$$

$$-\infty < x < \infty,$$

with conditions at infinities

$$w(-\infty) = 0, \quad w(\infty) = 1. \qquad (18)$$

Assuming $b > a$ we can translate the solution in such a way that

$$w(0) = a, \quad w(x_0) = b, \quad x_0 > 0. \qquad (19)$$

Then (17) takes the form

$$D_{|x|}^{\gamma} w + cw' - (k_1 + k_2)w + k_1 H(x) + k_2 H(x - x_0) = 0. \qquad (20)$$

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Applying the Fourier transform to (20), we obtain

$$-(|q|^\gamma - ciq + k_1 + k_2)\widetilde{w} + k_1\left(\pi\delta(q) - \frac{i}{q}\right)$$

$$+k_2\left(\pi\delta(q) - \frac{i}{q}\right)e^{-iqx_0} = 0,$$

so that

$$w(x) =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k_1(\pi\delta(q) - i/q) + k_2(\pi\delta(q) - i/q)e^{-iqx_0}}{|q|^\gamma - ciq + k_1 + k_2}e^{iqx}\,dq$$

$$= \frac{1}{2} - \frac{i}{2\pi}V.P.\int_{-\infty}^{\infty}\frac{k_1 + k_2 e^{-iqx_0}}{q(|q|^\gamma - ciq + k_1 + k_2)}e^{iqx}\,dq.$$

The first condition in (19) gives

$$a = w(0) = \frac{1}{2} - \frac{i}{2\pi}V.P.\int_{-\infty}^{\infty}\frac{k_1 + k_2 e^{-iqx_0}}{q(|q|^\gamma - ciq + k_1 + k_2)}\,dq$$

$$= \frac{1}{2} + \frac{\alpha w_*}{\pi}\int_0^{\infty}\frac{ds}{(s^\gamma + 1)^2 + \alpha^2 s^2}$$

$$+ \frac{1 - w_*}{\pi}\int_0^{\infty}\frac{\alpha s\cos(sx_1) - (s^\gamma + 1)\sin(sx_1)}{s[(s^\gamma + 1)^2 + \alpha^2 s^2]}\,ds,$$

where

$$\alpha = c(k_1 + k_2)^{1/\gamma - 1}, \quad x_1 = x_0(k_1 + k_2)^{1/\gamma}.$$

Next, using for $x_1 > 0$ the representation

$$\int_0^{\infty}\frac{(s^\gamma + 1)\sin(sx_1)}{s[(s^\gamma + 1)^2 + \alpha^2 s^2]}\,ds =$$

$$\int_0^{\infty}\frac{\sin(sx_1)}{s}\,ds - \int_0^{\infty}\frac{[s^{\gamma-1}(s^\gamma + 1) + \alpha^2 s]\sin(sx_1)}{(s^\gamma + 1)^2 + \alpha^2 s^2}\,ds$$

$$= \frac{\pi}{2} - \int_0^{\infty}\frac{[s^{\gamma-1}(s^\gamma + 1) + \alpha^2 s]\sin(sx_1)}{(s^\gamma + 1)^2 + \alpha^2 s^2}\,ds,$$

which yields an integral that goes to zero as $x_1 \to \infty$, we finally obtain

$$\frac{a - \frac{1}{2}w_*}{w_*} = F(\alpha, \gamma) + \frac{1 - w_*}{w_*}H(\alpha, \gamma, x_1), \tag{21}$$

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where

$$H(\alpha, \gamma, x) =$$

$$\frac{1}{\pi} \int_0^\infty \frac{\alpha \cos(sx) + [s^{\gamma-1}(s^\gamma + 1) + \alpha^2 s] \sin(sx)}{(s^\gamma + 1)^2 + \alpha^2 s^2} ds$$

and F as defined in (8).

The function $H(\alpha, \gamma, x)$ is a monotonically increasing function of α, $-\infty < \alpha < \infty$ for $x > 0$ and $1 \leq \gamma \leq 2$. It monotonically decreases with $x > 0$ for any α, $-\infty < \alpha < \infty$ and γ, $1 \leq \gamma \leq 2$ (see Figures 7, 8).
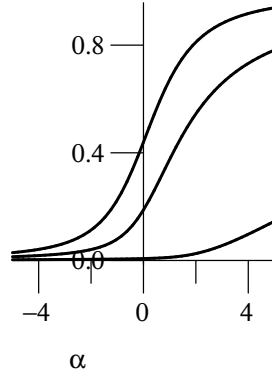

Figure 7: The dependence of the function H on α for $\gamma = 3/2$ and $x = 0.1, 1, 10$ (the larger is the x, the lower is the graph).

In addition,

$$H(-\infty, \gamma, x) = 0, \quad H(\infty, \gamma, x) = 1,$$

$$H(\alpha, \gamma, 0) = F(\alpha, \gamma) + \frac{1}{2}, \tag{22}$$

$$H(\alpha, 2, x) = \frac{1}{2}\left(1 + \frac{\alpha}{\sqrt{\alpha^2 + 4}}\right) \exp\left(-\frac{x}{2}(\sqrt{\alpha^2 + 4} - \alpha)\right).$$

The second condition in (19) gives

$$\frac{b - \frac{1}{2}(1 + w_*)}{1 - w_*} = F(\alpha, \gamma) - \frac{w_*}{1 - w_*} H(-\alpha, \gamma, x_1). \tag{23}$$

Equations (21) and (23) determine α and x_1 which are the scaled propagation velocity and distance between the fronts. Consider first equation
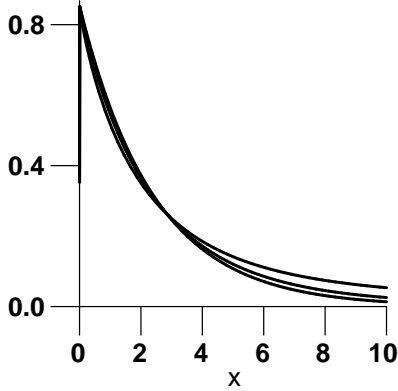
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Figure 8: The dependence of the function H on x for $\alpha = 2$ and $\gamma = 2, 3/2, 1$ (for large x, the larger is the γ, the lower is the graph).

(21). Since the last term on the right-hand side is positive and $F(\alpha, \gamma)$ is a monotonically increasing function of α, we see that the α determined by this equation is necessarily smaller than the α determined by the equation

$$\frac{a - \frac{1}{2}w_*}{w_*} = F(\alpha, \gamma),$$

which gives the speed of the $[0, w_*]$-wave (see (9)). In a similar way, considering equation (23) we conclude that the speed of the wave must be necessarily larger than that of the $[w_*, 1]$-wave. Combining these two results together, we see that if the $[0, 1]$-wave exists, then the speed of the $[0, w_*]$-wave is larger than the speed of $[w_*, 1]$-wave. We now show that this necessary condition is also a sufficient condition.

Consider first equation (21). Let us treat this equation as an equation for α with x_1 being a parameter. We will denote the solution as $\alpha = \alpha_1(x_1)$. For $x_1 = \infty$, we have $H = 0$, so that the equation reduces to

$$\frac{a - \frac{1}{2}w_*}{w_*} = F(\alpha_1(\infty), \gamma), \tag{24}$$

which yields the propagation speed of the $[0, w_*]$-wave. For $x_1 = 0$ using

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(22), we obtain that $\alpha_1(0)$ satisfies

$$a - \frac{1}{2} = F(\alpha_1(0), \gamma). \tag{25}$$

Since the right-hand side of equation (21) is an increasing function of α and a decreasing function of x_1, the solution $\alpha = \alpha_1(x_1)$ of this equation exists, is unique and is a monotonically increasing function for $0 < x_1 < \infty$ that varies between $\alpha_1(0)$ and $\alpha_1(\infty)$ that are determined by (25) and (24), respectively. Next, consider equation (23). Since the right-hand side of this equation is an increasing function of both α and x_1, the solution $\alpha = \alpha_2(x_1)$ exists, is unique and is a monotonically decreasing function for $0 < x_1 < \infty$ that varies between $\alpha_2(0)$ and $\alpha_2(\infty)$ that are determined by

$$b - \frac{1}{2} = F(\alpha_2(0), \gamma). \tag{26}$$

and

$$\frac{b - \frac{1}{2}(1 + w_*)}{1 - w_*} = F(\alpha_2(\infty), \gamma),$$

respectively. The graphs of functions $\alpha_1(x_1)$ and $\alpha_2(x_1)$ have exactly one intersection because $\alpha_2(0) > \alpha_1(0)$ (since F is monotonically increasing and $b > a$ – see equations (25), (26)) and $\alpha_1(\infty) > \alpha_2(\infty)$ (since these velocities are the velocities of the $[0, w_*]$-wave and the $[w_*, 1]$-wave for which this inequality is assumed).

Finally, the necessary and sufficient condition for the $[0, 1]$ wave to exist is that the velocity of the $[0, w_*]$ wave is larger than the velocity of the $[w_*, 1]$ wave.

5. FitzHugh-Nagumo equations

The FitzHugh-Nagumo system [45, 46] has been studied in many works as a simplified model of nerve conduction. The equations have the form

$$v_t = v_{\xi\xi} + f(v) - w, \quad w_t = \epsilon(v - bw),$$

where ϵ and b are positive parameters and $f(v)$ is, in general, a function with three zeros, two of which are stable, say $v = 0$, $v = 1$, and the third one $v = a$, $0 < a < 1$, is unstable. The system has a critical point $v = w = 0$, which is referred to as the rest state. In addition, for some parameter ranges

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there is another critical point $v = v_*$, $w = w_*$, referred to as the excited state. The problem is known to have traveling wave solutions that connect the two states as well as various pulse solutions [47, 48]. Moreover, many solutions can occur for the same parameter values.

In this paper we consider a generalization of the FitzHugh-Nagumo system that accounts for superdiffusion. We take $f(v) = -v + H(v - a)$ [48, 49] and restrict the consideration to traveling waves which connect the rest and the excited states, i.e., we do not consider pulse solutions. The problem written in the coordinate system that moves together with the wave has the form

$$D_{|x|}^{\gamma} v + cv' - v + H(v - a) - w = 0,$$

$$cw' + \epsilon v - b\epsilon w = 0, \quad -\infty < x < \infty,$$

with conditions at infinities

$$v(-\infty) = w(-\infty) = 0, \quad v(\infty) = v_* \equiv \frac{b}{1 + b},$$

$$w(\infty) = w_* \equiv \frac{1}{1 + b}.$$

Here the prime denotes the derivative with respect to the traveling wave coordinate $x = \xi - ct$, where c is the (unknown) propagation velocity. We remark that in order for the excited state to exist, one needs

$$a < \frac{b}{b + 1},$$

which we assume. Translating the wave in x so that $v(0) = a$ we can replace $H(v - a)$ by $H(x)$. Then applying the Fourier transform to both equations, we obtain

$$(-|q|^{\gamma} + ciq - 1)\widetilde{v} + \pi\delta(q) - \frac{i}{q} - \widetilde{w} = 0,$$

$$ciq\widetilde{w} + \epsilon\widetilde{v} - b\epsilon\widetilde{w} = 0.$$

Solving the equations for $\widetilde{v}$ and $\widetilde{w}$ and inverting the Fourier transform of the

solution yields

$$v(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi\delta(q) - i/q}{|q|^\gamma - ciq + 1 + \epsilon/(b\epsilon - ciq)} e^{iqx} \, dq$$

$$= \frac{b}{2(1+b)}$$

$$+ \frac{1}{\pi} \mathrm{Im} \int_0^{\infty} \frac{e^{iqx}}{q(q^\gamma - ciq + 1 + \epsilon/(b\epsilon - ciq))} \, dq$$

$$= \frac{b}{2(1+b)} +$$

$$\frac{1}{\pi} \int_0^{\infty} \frac{cq\,(1 - \epsilon_1)\cos(qx) + (q^\gamma + 1 + \epsilon_2)\sin(qx)}{q\left[(q^\gamma + 1 + \epsilon_2)^2 + c^2q^2\,(1 - \epsilon_1)^2\right]} \, dq,$$

where

$$\epsilon_1 = \frac{\epsilon}{b^2\epsilon^2 + c^2q^2}, \qquad \epsilon_2 = \frac{b\epsilon^2}{b^2\epsilon^2 + c^2q^2}.$$

Evaluating this equation at $x = 0$ yields

$$\frac{a - \frac{1}{2}v_*}{v_*} = P(c, \gamma, b, \epsilon), \tag{27}$$

where

$$P(c, \gamma, b, \epsilon) =$$

$$\frac{1 + b}{\pi b} \int_0^{\infty} \frac{c\,(1 - \epsilon_1)\, dq}{(q^\gamma + 1 + \epsilon_2)^2 + c^2q^2\,(1 - \epsilon_1)^2}.$$

The dependence of function P on the propagation speed c may exhibit a complex nonmonotonic behavior that is illustrated in Figure 9. Function P is an odd function of c that goes to $\pm 1/2$ as $c \to \pm\infty$. For relatively large ϵ, the function is monotonic. As ϵ decreases, the function becomes nonmonotonic (at $\epsilon = 1/b^2$). The smaller is the ϵ, the more pronounced the extrema are. In the limit $\epsilon \to 0$, the function P tends to a function of c that is monotonically increasing for $c < 0$ and for $c > 0$ with a discontinuity at $c = 0$. The limits as $c \to \mp 0$ are $\pm 1/(2b)$.

Figures 10 and 11 depict the v component of the solution for specific parameter values as a function of the spatial variable. The value of the propagation velocity is found from (27).

Figure 12 shows the partition of the parameter space into regions with different number of solutions for $0 < a < 1/2$. Here in (n, m), n is the number of rest-state-dominated waves and m is the number of excited-state-dominated waves. If b is less than the critical value $a/(1-a)$, no fronts exist because there is no excited state. There are two regions in the (b, ϵ) plane marked $(1, 2)$. In the lower one the existence of two excited-state-dominated waves is unconditional, i.e., for any value of a. In the upper one, a must be sufficiently small for the two solutions to exist. If $a > 1/2$ the diagram is similar, with the only difference that the lower, unconditional region $(1, 2)$ disappears. If $a < 0$, the behavior is similar in the sense that (n, m) regions become (m, n) regions because the function P is an odd function. Finally, this parameter diagram is qualitatively the same for any γ, $1 < \gamma < 2$. A quantitative difference is that with the decrease of γ parameter regions where multiple solutions exist become smaller.
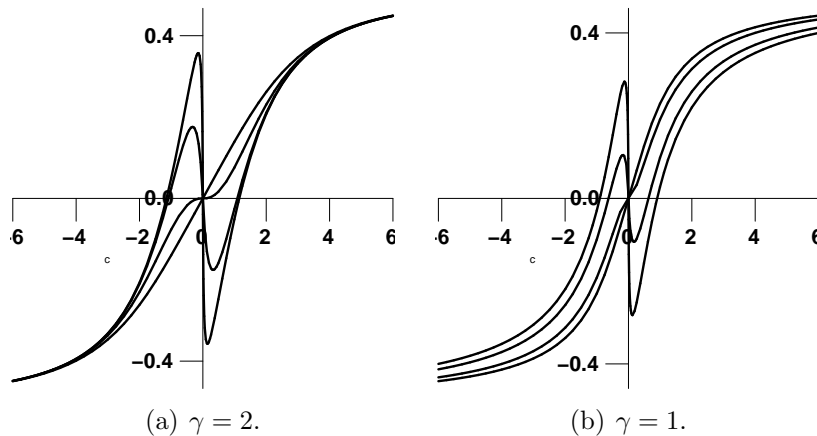

(a) $\gamma = 2$.　　　　　(b) $\gamma = 1$.

Figure 9: The dependence of the function P on c for $b = 1$, $\epsilon = 5.0, 1.0, 0.1, 0.01$ and two values of γ. The smaller is the ϵ, the more pronounced the extrema of the functions are.

6. Conclusion

We study front propagation in reaction-superdiffusion problems. Specifically, we consider traveling wave propagation, domain pinning, and systems of waves governed by a bistable single equation as well as FitzHugh-Nagumo equations. The reaction terms in these equations are taken in the form of piecewise linear functions, which allows us to determine exact solutions of
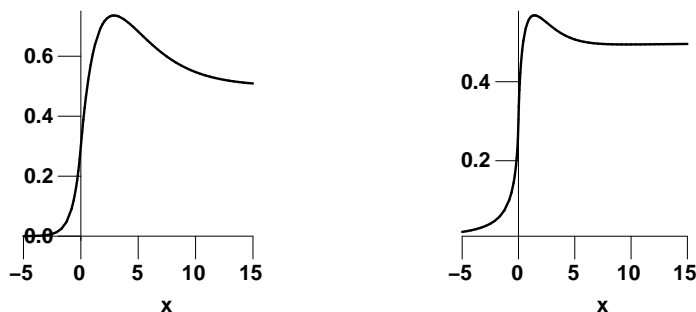
(a) $\gamma = 2$, $c = -0.74$. (b) $\gamma = 1$, $c = -0.241$.

Figure 10: Excited-state-dominated wave (goes to the left). The v-profile of the solution is shown as a function of x for two values of γ. Here $a = 0.3$, $b = 1$, $\epsilon = 0.1$ and the propagation velocity c is determined by equation (27).

the problems. We discuss the effect of superdiffusion on front propagation. The main conclusions are as follows.

- If one state significantly dominates the other, then superdiffusion results in much faster displacement of the weaker phase than regular diffusion.

- If two states are of comparable strength, superdiffusion is not noticeable.

- Superdiffusive systems of waves obey the same rules as in the case of regular diffusion.

- Domain pinning occurs more readily for superdiffusion than for normal diffusion. In fact, pinning becomes inevitable in the limit $\gamma \to 1$.

- In the FitzHugh-Nagumo system, superdiffusion reduces the parameter range of multiple solutions.

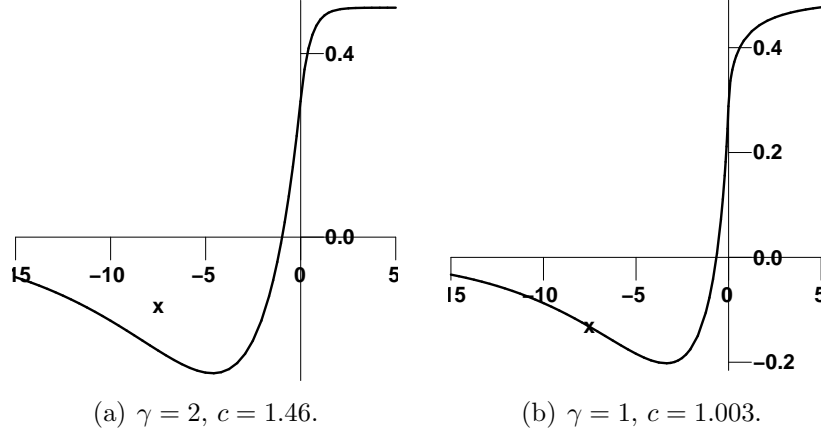(a) $\gamma = 2$, $c = 1.46$.   (b) $\gamma = 1$, $c = 1.003$.

Figure 11: Rest-state-dominated wave (goes to the right). The v-profile of the solution as a function of x for two values of γ. Here $a = 0.3$, $b = 1$, $\epsilon = 0.1$ and the propagation velocity c is determined by equation (27).

A. Properties of the function $a(c; \gamma)$

Hereinafter the monotonicity of the relation $a(c)$ is proved for the problem

$$\mathfrak{D}^{\gamma}_{|x|} w + c \frac{dw}{dx} - w + H(x) = 0, \tag{A.1}$$

where the source term (4) was used, and the dependence on the parameter k was removed by scaling (1) with $t \mapsto k\,t$ and $x \mapsto k^{1/\gamma}\,x$. The fractional operator is defined in a more general way as a weighted sum of leftward and rightward fractional derivatives in Fourier space:

$$\mathcal{F}_{x \to q} \left\{ \mathfrak{D}^{\gamma}_{|x|} w(x) \right\} =$$

$$- \sec\left(\frac{\pi\gamma}{2}\right) \left(p(iq)^{\gamma} + r(-iq)^{\gamma}\right) \mathcal{F}_{x \to q} \left\{ w(x) \right\},$$

so that

$$r = 1 - p, \quad p = \frac{1 + \nu}{2}, \quad r = \frac{1 - \nu}{2}, \quad -1 \leqslant \nu \leqslant 1.$$

Then $\nu = 0$ conforms to Riesz derivative and $\nu = 1$ – to Weyl derivative. The negative range of ν corresponds to derivatives with more weight on the
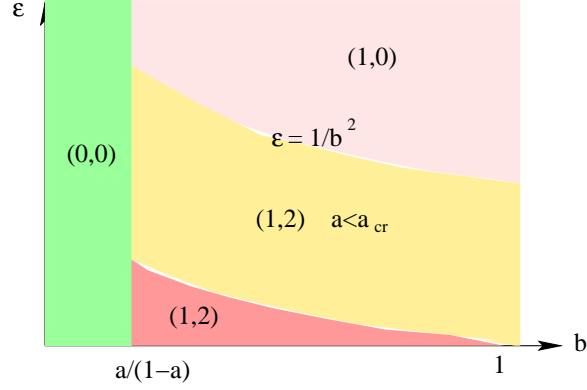
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Figure 12: Parameter regions with different number of solutions. See text.

leftward direction and is quite superfluous, as any problem with a dominating leftward propagation direction can be equivalently formulated with a rightward one. An equivalent form to be used below is

$$\mathcal{F}_{x \to q} \left\{ \mathfrak{D}_{|x|}^{\gamma} w(x) \right\} = -|q|^{\gamma} - i \operatorname{sgn}(q)\, \mu\, |q|^{\gamma}, \quad \mu = \nu \tan\frac{\pi\gamma}{2}.$$

The relation connecting the propagation velocity to the kinetics parameter a for (A.1) is obtained similarly to (8) and reads

$$a(c; \gamma, \nu) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{c - \mu\, q^{\gamma-1}}{(1 + q^\gamma)^2 + (c - \mu\, q^{\gamma-1})^2\, q^2}\, dq.$$

For $\nu = 0$ the relation simplifies to a form equivalent to (8):

$$a(c; \gamma, 0) = \frac{1}{2} + \frac{c}{\pi} \int_0^\infty \frac{dq}{(1 + q^\gamma)^2 + (cq)^2}.$$

For $\gamma = 2$ the value of ν is immaterial and

$$a(c; 2, \nu) - \frac{1}{2} = \frac{c}{\pi} \int_0^\infty \frac{dq}{(1 + q^2)^2 + (cq)^2} =$$

$$\frac{c}{\pi(b_2^2 - b_1^2)} \int_0^\infty \left(\frac{1}{q^2 + b_1^2} - \frac{1}{q^2 + b_2^2} \right) dq$$

with $b_1, b_2 > 0$ satisfying

$$b_1^2 b_2^2 = 1, \quad b_1^2 + b_2^2 = c^2 + 2$$

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by which
$$b_1 b_2 = 1, \quad (b_1 + b_2)^2 = c^2 + 4.$$

Then
$$a(c; 2, \nu) - \frac{1}{2} = \frac{c}{2\left(b_2^2 - b_1^2\right)} \left(\frac{1}{b_1} - \frac{1}{b_2}\right) =$$

$$= \frac{c}{2b_1 b_2 (b_1 + b_2)} = \frac{c}{2\sqrt{c^2 + 4}}.$$

The same result might be obtained directly through the ordinary differential equation
$$u'' + cu' - u + H(x) = 0.$$

For the general function $a(c; \gamma, \nu)$ the behavior at $c \to \pm\infty$ is as follows. Defining $q' = qc$, $c > 0$
$$a - \frac{1}{2} \sim$$

$$\sim \frac{1}{\pi} \int_0^\infty \frac{dq'}{1 + q'^2}$$

$$+ \frac{c^{-\gamma}}{\pi} \int_0^\infty \frac{q'^{\gamma-1}\left[\mu\left(q'^2 - 1\right) - 2q'\right]}{\left(1 + q'^2\right)^2} dq' + o\left(c^{-\gamma}\right)$$

$$= \frac{1}{2} + \frac{(\gamma - 1)(\nu + 1)}{2\cos(\pi\gamma/2)} c^{-\gamma} + o\left(c^{-\gamma}\right), \quad c \to \infty. \tag{A.2a}$$

Similarly, for $c \to -\infty$
$$a - \frac{1}{2} =$$

$$-\frac{1}{2} + \frac{(\gamma - 1)(\nu + 1)}{2\cos(\pi\gamma/2)} |c|^{-\gamma} + o\left(|c|^{-\gamma}\right), \quad c \to -\infty. \tag{A.2b}$$

Therefore the slope at the tails is always positive.

Obviously, $a(c; \gamma, \nu)$ is an odd function of c if $\gamma = 2$ or $\nu = 0$. For other values of γ and ν the integrand denominator is positive throughout, and thus the numerator must change sign for some $q_\star > 0$ at $a = 1/2$, as a is a continuous function of c and such a point exists by the intermediate value theorem:
$$\exists\, q_\star \; : \; q_\star^{\gamma-1} = \frac{c}{\mu} = \frac{c}{\nu}\cot\frac{\pi\gamma}{2}.$$

Referring to the range $0 < \nu \leqslant 1$ and noting that $\cot(\pi\gamma/2) < 0 \; \forall \; 1 < \gamma < 2$, an immediate conclusion is that the point $a = 1/2$ is obtained at $c < 0$.

For the negative range of ν $c > 0$. Figure 13 depicts the function $a(c; \gamma, \nu)$ for $\gamma = 1.2$ and several values of ν. The behavior bears no qualitative changes throughout the range $1 < \gamma < 2$.
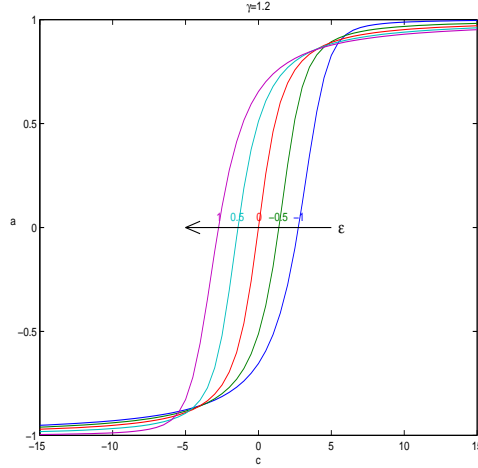

Figure 13: Relation $a(c; \gamma, \nu)$ for $\gamma = 1.2$ and several values of ν.

It is possible to evaluate the lower and upper bounds of the slope. Differentiating the function $a(c; \gamma, \nu)$ with respect to c,

$$\frac{da}{dc} = \frac{1}{\pi} \int_0^\infty \frac{(1 + q^\gamma)^2 - (qc - \mu\, q^\gamma)^2}{[(1 + q^\gamma)^2 + (qc - \mu\, q^\gamma)^2]^2} dq. \tag{A.3}$$

Defining

$$A(q; \gamma) \stackrel{def}{=} 1 + q^\gamma, \quad B(q; \gamma, c, \nu) \stackrel{def}{=} qc - \mu q^\gamma,$$

$$h(q; \gamma, c, \nu) = \frac{B}{A},$$

the integrand is expressed as

$$I(q; \gamma, c, \nu) = \frac{A^2 - B^2}{(A^2 + B^2)^2} = \frac{1 - h^2}{A^2(1 + h^2)^2}.$$

The function

$$f(h^2) = \frac{1 - h^2}{(1 + h^2)^2}$$

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satisfies $-1/8 \le f \le 1$ for $h^2 \ge 0$. Thus

$$-\frac{1}{8} \int_0^\infty \frac{dq}{A^2} \le \pi \frac{da}{dc} \le \int_0^\infty \frac{dq}{A^2}.$$

Computing the integral as

$$\int_0^\infty \frac{dq}{A^2} = \frac{\pi(\gamma-1)}{\gamma^2 \sin(\pi/\gamma)} \le 1, \; 1 \le \gamma \le 2$$

we obtain

$$-\frac{1}{8\pi} \le \frac{da}{dc} \le \frac{1}{\pi}$$

regardless of all parameter values.

In fact, the function $a(c; \gamma, \nu)$ should be monotone in c, however it is not seen directly from (A.3). To show monotonicity the expression for the difference $a_2 - a_1$ is derived:

$$\pi(a_2 - a_1) = (c_2 - c_1) \int_0^\infty \frac{A^2 - B_1 B_2}{(A^2 + B_1^2)(A^2 + B_2^2)} dq,$$

$$B_i \equiv B(q; \gamma, c_i, \nu), \; i = \{1, 2\}.$$

Suppose that there exists a point where da/dc changes sign (not just vanishes, but becomes negative, so that the point is a real extremum and not an inflection point). Then by continuity and the general ascending behavior of a (see equations (A.2)) there must be another extremum. Furthermore, the first one should be a maximum and the second one – a minimum. If there are any more extrema, they should come in such pairs. Therefore it is possible to take three points that satisfy

$$c_1 < c_2 < c_3 \quad : \quad a(c_1) = a(c_2) = a(c_3),$$

i.e. c_1 is located before the maximum, c_2 – between the two extrema and c_3 – after the minimum. Evaluating the derivative at these three points, the following must hold

$$\int_0^\infty \frac{A^2 - B_1^2}{(A^2 + B_1^2)^2} dq > 0,$$

$$\int_0^\infty \frac{A^2 - B_2^2}{(A^2 + B_2^2)^2} dq < 0, \quad \int_0^\infty \frac{A^2 - B_3^2}{(A^2 + B_3^2)^2} dq > 0. \tag{A.4}$$

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The function B is linear in c and thus

$$B_1^2 < B_2^2 < B_3^2.$$

Then at every integration point q

$$\frac{A^2 - B_2^2}{(A^2 + B_2^2)^2} > \frac{A^2 - B_3^2}{(A^2 + B_2^2)^2} > \frac{A^2 - B_3^2}{(A^2 + B_3^2)^2}.$$

Therefore it is impossible that both inequalities in (A.4) hold simultaneously. Hence such a non-monotonous behavior cannot exist, and the lower bound can be modified to a weak positiveness

$$0 \leqslant \frac{da}{dc} \leqslant \frac{1}{\pi}.$$

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