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Turing instability of anomalous reaction–anomalous diffusion systems

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Linear stability theory is developed for an activator–inhibitor model where fractional derivative operators of generally different exponents act both on diffusion and reaction terms. It is shown that in the short wave limit the growth rate is a power law of the wave number with decoupled time scales for distinct anomaly exponents of the different species. With equal anomaly exponents an exact formula for the anomalous critical value of reactants diffusion coefficients' ratio is obtained.

1 Introduction

Reaction–diffusion equations have been used for a long time to model numerous natural phenomena, far beyond the immediate chemical application. A remarkable property of systems governed by these equations is the onset of a short wave (Turing) instability leading to a spontaneous breakdown of the translational invariance [14].

In the past decade a plethora of transport phenomena not amenable to modelling by standard Brownian motion and conventional diffusion equation has been discovered. These *anomalous* diffusive processes are characterised by temporal scaling of the mean square displacement of the type $\langle r^2(t)\rangle \sim t^\gamma$, where $0 < \gamma < 1$ (sub-diffusion) or $\gamma > 1$ (super-diffusion). The anomalous scaling has been theoretically predicted for diffusion in fractals and disordered media (refer to the book in [2] and review papers [12, 13]) and studied in numerous experiments. Sub-diffusion has been observed in porous media [3], in glass-forming systems [23], in cell membranes [18], inside living cells [24] as well as in many other physical and biological systems. An essential progress in understanding and mathematical modelling has been achieved. It is found that sub-diffusive processes can be modelled by a memory term containing a fractional derivative.

The notion of a fractional derivative enables differentiation and integration to an arbitrary order through generalisation of Cauchy's formula and analytic continuation of the Γ function. An integral of order γ,

$$\mathrm{I}^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t f(\tau)(t-\tau)^{\gamma-1} d\tau, \; t > 0, \; \gamma \in \mathbb{R}, \tag{1.1}$$

along with the fact that the differentiation operator is the left-inverse but not the right-inverse of the integral operator, lead to the definition of a derivative of order γ

as

$$\mathrm{D}^\gamma f(t) = \mathrm{D}^m \mathrm{I}^{m-\gamma} f(t) = \frac{1}{\Gamma(m-\gamma)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^{\gamma+1-m}},$$
$$m-1 < \gamma \leqslant m, \; m \in \mathbb{N}. \tag{1.2}$$

Formal substantiation of the definitions, rules of fractional calculus and proof of consistence with the basic calculus theorems can be found in the book [16]. From a physics standpoint derivatives of arbitrary order grant a dynamical system memory, as the kernel $(t-\tau)^{\gamma+1-m}$ convolves the system state from $t = 0$ and onwards. In the present context such a memory mechanism enables an essential hindrance of the molecular motion due to specific properties of the medium and leads to the understanding that normal diffusion is a special limit of a whole family of processes; though widely used, it is insufficient in many cases.

Fractional order differentiation is closely related to the continuous time random walk, where the time period between consecutive jumps is given by a probability function. For $0 < \gamma < 1$ this density attentuates faster than the Gaussian distribution and possesses a cusp at $x = 0$, yielding an essentially slower dispersion of particles, alias *sub-diffusion*. One example is the generalised Fokker–Planck equation

$$\frac{\partial^\gamma}{\partial t^\gamma} P(\mathbf{r}, t) = \kappa_\gamma \nabla^2 P(\mathbf{r}, t), \tag{1.3a}$$

or equivalently

$$\frac{\partial}{\partial t} P(\mathbf{r}, t) = \kappa_\gamma \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}} \nabla^2 P(\mathbf{r}, t). \tag{1.3b}$$

If a group of particles is released at the origin at time $t = 0$ and their dispersion is traced, the time-dependent distribution is Gaussian with a growing variance for $\gamma = 1$, and a cusped function for $0 < \gamma < 1$. The cusp gradually disappears as the uniform dispersion equilibrium is approached. Numerous types of fractional differential equations, their solutions and the corresponding physical interpretation can be found in the reviews [12, 13] and the book [17].

Reaction processes are observed in a number of anomalous systems, for example recombination in sub-diffusive media [19, 20], transport processes by formation of carious lesion of the enamel [10] and spreading of tumour cells [4]. Standard reaction kinetics derived from the law of mass action [7] is inapplicable both for diffusion-limited reactions, where the reaction term is subject to memory effects and has to be modified by application of a fractional derivative [25], and for activation-limited reactions, modelled by an integro-differential equation [21]. Thus, even the functional form of the reaction term is not universal, and is determined by the reaction type and underlying physical factors.

The investigation of instability in anomalous reaction–diffusion systems could provide a test for mathematical models used for their description. The present paper analyses the onset of Turing instability for diffusion-limited reaction, suitable for modelling by a fractional derivative. In general, it is not obvious that the reaction anomaly exponent coincides with that of the diffusion. Moreover, each species may diffuse with its own anomaly exponent. As confirmed by a number of experiments, diffusion within a cell cytoplasm or its organelles is often anomalous and reactions are diffusion-limited [5]. The diffusing molecules differ greatly in size, spatial structure, polarity, binding energy, escape time and other properties affecting their progress through hydrophilic and hydrophobic regions of the cell. Thus the expected differences in diffusion and reaction anomaly exponents are quite natural. One example of size effect is the change in exponent value due to different ratio of bead probe diameter to filament size in actin networks [1].

Hence, it is of interest to create a model embracing all possible exponent combinations and elucidate the connection between the few special cases studied hitherto. Anomalous diffusion with normal reaction and identical anomaly exponents for all species, i.e., equally slowed dispersion of all reactants, was studied in [7, 8], for the first time describing oscillatory unstable modes that never appear in normal diffusion. The new stability threshold due to these modes was found in [15] and the notion of diffusion coefficients' ratio was further extended to the case of unequal anomaly exponents, i.e., essentially differing scales of particles dispersion for each species. In [6] numerical simulations of patterns of this type are presented. On the other hand, the simplest model with diffusion and reaction both anomalous (identical memory terms for all components) predicted unchanged stability criterion [11]. To paraphrase, when both processes are slowed down to the same extent, the stability threshold remains normal. With the new model the earlier results [6–8, 11, 15] are recovered in particular limits and the transition between these limit cases is described.

2 Mathematical model

A two-species activator–inhibitor system evolves according to

$$\frac{\partial}{\partial t}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = C\nabla^2 \begin{pmatrix} \mathscr{D}^{1-\gamma_1} n_1 \\ \mathscr{D}^{1-\gamma_2} n_2 \end{pmatrix} + \begin{pmatrix} \mathscr{D}^{1-\delta_1} f_1 \\ \mathscr{D}^{1-\delta_2} f_2 \end{pmatrix} \quad \text{in } \Omega, \tag{2.1}$$

$$\mathbf{n}(\mathbf{r}, t) = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad \mathbf{f}(\mathbf{n}) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix},$$

where n_1 and n_2 are respectively the activator and inhibitor density numbers, depending on position $\mathbf{r}$ and time t. Here $\mathbf{f}(\mathbf{n}(\mathbf{r}, t))$ denotes the reaction kinetics. The matrix of diffusion coefficients is taken diagonal and constant to isolate the anomaly effects, with d being the ratio of the diffusion coefficients. The fractional operator is defined as follows [7, 8]:

$$\mathscr{D}^{1-\gamma} = \mathfrak{D}^{1-\gamma} + \mathscr{L}^{-1}(\mathfrak{D}^{-\gamma}[\,\cdot\,]_{t=0}), \tag{2.2}$$

where the operator

$$\mathfrak{D}^{-\gamma} y(t) = \frac{d^{-\gamma} y(t)}{dt^{-\gamma}} = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{y(\tau)}{(t-\tau)^{1-\gamma}}\, d\tau \tag{2.3}$$

denotes the Riemann-Liouville fractional integral and

$$\mathfrak{D}^{1-\gamma} y(t) = \frac{d^{1-\gamma} y(t)}{dt^{1-\gamma}} = \frac{d}{dt} \frac{d^{-\gamma} y(t)}{dt^{-\gamma}}; \tag{2.4}$$

$\mathscr{L}^{-1}$ denotes the inverse Laplace transform. The regularisation term $\mathscr{L}^{-1}(\mathfrak{D}^{-\gamma}[\,\cdot\,]_{t=0})$ arises due to the manner initial conditions are handled in Laplace transform formalism, i.e., the function $\mathbf{n}(\mathbf{r},t)$ is discontinuous at $t = 0$ in the sense that $\lim_{t\to 0^-}\mathbf{n}(\mathbf{r},t) = 0$, whereas $\lim_{t\to 0^+}\mathbf{n}(\mathbf{r},t)$ should equal the prescribed initial function $\mathbf{n}(\mathbf{r},0) = \mathbf{n}|_{t=0}$ (see [7] for a detailed derivation). The exponents $0 < \gamma_j, \delta_j < 1$, $j \in \{1,2\}$, are generally different for each species. For the sake of simplicity, the domain Ω is the whole space or has a rectangular shape,

$$\Omega \subset \mathbb{R}^p, \quad p \in \{1,2,3\}. \tag{2.5}$$

Boundary conditions are assumed periodic or zero flux across the domain boundary $\nabla\mathbf{n}\cdot\nu = 0$ on $\partial\Omega$ (ν is the outward normal). Distinct anomaly exponents grant the processes of diffusion and reaction the ability to evolve on different time scales.

3 Instability criterion

Temporal evolution of a small perturbation $\Delta\mathbf{n}$ about a uniform steady-state $\mathbf{n}_0$, $\mathbf{f}(\mathbf{n}_0) = 0$ is governed by the linearised system

$$\frac{\partial}{\partial t}\begin{pmatrix}\Delta n_1\\ \Delta n_2\end{pmatrix} = C\nabla^2\begin{pmatrix}\mathscr{D}^{1-\gamma_1}\Delta n_1\\ \mathscr{D}^{1-\gamma_2}\Delta n_2\end{pmatrix} + \nabla\mathbf{f}\begin{pmatrix}\mathscr{D}^{1-\delta_1}\Delta n_1\\ \mathscr{D}^{1-\delta_2}\Delta n_2\end{pmatrix} \tag{3.1}$$

wherein

$$\nabla\mathbf{f}_{jk} = \left(\frac{\partial f_j}{\partial n_k}\right)_{\mathbf{n}_0} \tag{3.2}$$

is the kinetic sensitivity matrix. It is assumed that the steady state $\mathbf{n}_0$ is stable in the absence of diffusion (and $\delta_1 = \delta_2 = 1$). Thus the eigenvalues of $\nabla\mathbf{f}$ must have negative real parts, and the entries of $\nabla\mathbf{f}$ satisfy $\mathrm{tr}\nabla\mathbf{f} < 0$, $\det\nabla\mathbf{f} > 0$. Later on, following [14], it is assumed that $\nabla f_{11} > 0$, $\nabla f_{22} < 0$ and $d > 1$.

Upon subsequent application of temporal Laplace transform (denoted by tilde) and spatial Fourier transform (denoted by hat), equation (3.1) becomes

$$\widehat{\widetilde{\Delta\mathbf{n}}} = S_{\mathscr{L}}^{-1}\widehat{\Delta\mathbf{n}}|_{t=0}, \tag{3.3}$$

where $S_{\mathscr{L}}$ is a 2×2 matrix with dependence on the Laplace and Fourier transform variables s and $\mathbf{q}\in\mathbb{R}^p$, $q = |\mathbf{q}|$:

$$S_{\mathscr{L}} = \begin{pmatrix} s + q^2 s^{1-\gamma_1} - \nabla f_{11}s^{1-\delta_1} & -\nabla f_{12}s^{1-\delta_1}\\ -\nabla f_{21}s^{1-\delta_2} & s + d\,q^2 s^{1-\gamma_2} - \nabla f_{22}s^{1-\delta_2}\end{pmatrix}. \tag{3.4}$$

The system dispersion relation is then given by

$$s^{2-\delta_1-\delta_2}\,D(q,s;\gamma_1,\gamma_2,\delta_1,\delta_2) = \det S_{\mathscr{L}} = 0. \tag{3.5}$$

If the domain Ω is infinite, appropriate decay of $\mathbf{n}(\mathbf{r},t)$ is assumed for $|\mathbf{r}|\to\infty$, and the spectrum $\mathbf{q}$ is continuous. Otherwise, both for zero-flux and periodic boundary conditions the spectrum is discrete.

In anomalous systems existence of roots with positive real part does not imply instability [8]. Therefore, further analysis of the disturbance temporal evolution is required to characterise the truly unstable roots. In the Fourier domain the perturbed density vector will have the form

$$\widehat{\Delta\mathbf{n}}(q,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} I(q,s)e^{st}\,ds, \tag{3.6a}$$

$$I(q,s) = S_{\mathscr{L}}^{-1}\,\widehat{\Delta\mathbf{n}}|_{t=0} = \frac{\mathscr{A}(q)s + \mathscr{B}(q) + \mathscr{C}_1(q) + \mathscr{C}_2(q)}{s^{2-\delta_1-\delta_2}D(q,s;\gamma_1,\gamma_2,\delta_1,\delta_2)}, \tag{3.6b}$$

$$\mathscr{A}(q) = \widehat{\Delta\mathbf{n}}|_{t=0}, \qquad \mathscr{B}_j(q) = q^2 s^{1-\gamma_j} C_{kk}\widehat{\Delta n_j}|_{t=0},$$
$$\mathscr{C}_{1j}(q) = s^{1-\delta_j}\nabla\mathbf{f}_{jk}\widehat{\Delta n_k}|_{t=0}, \quad \mathscr{C}_{2j}(q) = -s^{1-\delta_k}\nabla\mathbf{f}_{kk}\widehat{\Delta n_j}|_{t=0},$$
$$j,k \in \{1,2\},\ j \neq k, \tag{3.6c}$$

$$D(q,s;\gamma_1,\gamma_2,\delta_1,\delta_2) = s^{\delta_1+\delta_2} + d\,q^2 s^{\delta_1+\delta_2-\gamma_2} + q^2 s^{\delta_1+\delta_2-\gamma_1} - \nabla f_{22}s^{\delta_1} - \nabla f_{11}s^{\delta_2}$$
$$+ d\,q^4 s^{\delta_1-\gamma_1+\delta_2-\gamma_2} - \nabla f_{22}q^2 s^{\delta_1-\gamma_1} - d\nabla f_{11}q^2 s^{\delta_2-\gamma_2} + \det\nabla\mathbf{f}. \tag{3.6d}$$

The essential difference between (3.6) and its particular case with $\delta_1 = \delta_2 = 1$ treated in [8, 15] is the possible singularity at $s = 0$ in (3.6b). The latter can be eliminated by means of the variable transformation $\xi = s^{\varpi}$, ϖ being a constant constructed in such a way that the integrand has no singularity at $\xi = 0$. Later on ϖ will be taken rational. For example, when $\epsilon_j \overset{def}{=} \delta_j - \gamma_j > 0$, $j \in \{1,2\}$, the appropriate power is $\varpi = \min\{\delta_1,\delta_2\}$. To see that, note that the numerator and denominator lowest s powers are $z = 1 - \max\{\delta_1,\delta_2\}$ and $p = 2 - \delta_1 - \delta_2$ respectively. Upon taking $\varpi = 1 - (p - z)$ and changing the integration variable to ξ the integrand has no zero or branching point at $\xi = 0$. This property allows its straightforward transformation into a rational function and application of Watson's lemma later on.

If one or both ϵ_j are negative, the situation is more complicated. If $\epsilon_j < 0$, but $\epsilon_k > 0$, $p = 2 - \gamma_j - \delta_k$ $(j \neq k)$. As to z, it is different for the two species. $z_j = 1 - \max\{\delta_1,\delta_2\}$, as before, but

$$z_k = \begin{cases} 1-\gamma_j & \delta_k < \gamma_j \\ 1-\delta_k & \delta_k > \gamma_j \end{cases}, \quad j \neq k. \tag{3.7}$$

Therefore the transformation will be either $\xi_k = s^{\delta_k}$ or $\xi_k = s^{\gamma_j}$. Since the power ϖ participates in determination of the instability sector (see below), it turns out that the anomaly of the processes of species j may manifest itself in the stability characteristics of species k. If both $\epsilon_j, \epsilon_k < 0$, $p = 2 - \gamma_j - \gamma_k$ and

$$z_j = \begin{cases} 1-\gamma_j & \delta_k < \gamma_j \\ 1-\delta_k & \delta_k > \gamma_j \end{cases}, \quad j \in \{1,2\}. \tag{3.8}$$

Thus $\xi_j = s^{\gamma_k}$ or $\xi_j = s^{\gamma_j-\epsilon_k}$, and the interference of species in the stability characteristics of one another is mutual.

As mentioned above, for any combination of the anomaly exponents it is possible to remove the singularity of $I(q,s)$ at $s = 0$ by means of the variable transformation $\xi = s^{\varpi}$. It is reasonable to assume that all powers of s are reduced fractions, as the set of rational numbers is dense within $\mathbb{R}$. Then $I(q,s)$ becomes a ratio of two functions of rational

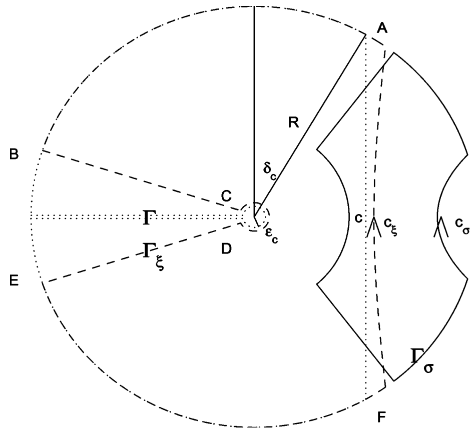

FIGURE 1. Integration contour Γ (dotted) and modified contours Γ_ξ (dashed), Γ_σ (solid) after successive transformations $\xi = s^{\varpi}$, $\sigma = \xi^{1/\mathscr{R}}$.

powers of s. The integration path is closed in order to use the residue theorem. The original and transformed contours are shown in Figure 1 as dotted and dashed curves respectively. The vertical line $\Re s = c$ becomes a curve, whose argument at infinite distance from origin equals $\pm\varpi\pi/2$, and the branch cut along the negative s axis becomes a sector of argument $[\varpi\pi, (2-\varpi)\pi]$.

Now the inversion integral can be evaluated as follows:

$$2\pi i\widehat{\Delta\mathbf{n}} = \int_{\Re s=c} I(q,s)e^{st}\,ds = \lim_{R\to\infty,\epsilon_c\to 0}\left(\oint_{\Gamma} I(q,s)e^{st}\,ds - \int_{\Gamma\setminus\{\Re s=c\}} I(q,s)e^{st}\,ds\right). \quad (3.9)$$

Using the zero/pole-eliminating transformation and replacing $I(q,s)\mapsto \mathscr{I}(q,\xi)$, one finds

$$\int_{\Re s=c} I(q,s)e^{st}\,ds = \int_{c_\xi} \mathscr{I}(q,\xi)e^{\xi^{1/\varpi}t}\,d\xi$$
$$= \lim_{R\to\infty,\epsilon_c\to 0}\left(\oint_{\Gamma_\xi} \mathscr{I}(q,\xi)e^{\xi^{1/\varpi}t}\,d\xi - \int_{\Gamma_\xi\setminus c_\xi} \mathscr{I}(q,\xi)e^{\xi^{1/\varpi}t}\,d\xi\right). \quad (3.10)$$

The integral along the arc CD vanishes in the limit $\epsilon_c \to 0$:

$$\lim_{\epsilon_c\to 0}\int_{\varpi\pi}^{-\varpi\pi} \mathscr{I}(q,\epsilon_c e^{i\theta})\exp\left(\epsilon_c^{1/\varpi}e^{i\theta/\varpi}t\right)\epsilon_c i e^{i\theta}\,d\theta = 0. \quad (3.11)$$

The integrals along the arcs AB and EF vanish in the limit $R \to \infty$. For the arc AB (and similarly for EF)

$$\lim_{R\to\infty}\left|\int_{\varpi\pi/2}^{\varpi\pi} \mathscr{I}(q,Re^{i\theta})\exp\left(R^{1/\varpi}e^{i\theta/\varpi}t\right)Rie^{i\theta}\,d\theta\right|$$
$$\leqslant \lim_{R\to\infty}\max_{\varpi\pi/2<\theta<\varpi\pi}|\mathscr{I}(q,Re^{i\theta})|R\int_{\varpi\pi/2}^{\varpi\pi}\exp\left(R^{1/\varpi}\cos(\theta/\varpi)t\right)d\theta. \quad (3.12)$$

Changing variables $\varphi = \theta/\varpi$ and denoting $\rho = R^{1/\varpi}$,

$$R\int_{\varpi\pi/2}^{\varpi\pi} \exp\left(R^{1/\varpi}\cos(\theta/\varpi)t\right) d\theta = \varpi\rho^{\varpi}\int_{\pi/2}^{\pi} \exp\left(\rho\cos(\varphi)t\right) d\varphi \leqslant K(t)\rho^{\varpi-1}. \quad (3.13)$$

For the proof of the last inequality see [8]. Combining with

$$\lim_{R\to\infty}\max_{\varpi\pi/2<\theta<\varpi\pi} |\mathscr{I}(q, Re^{i\theta})| = 0 \quad (3.14)$$

yields the proposed result.

The integrals along the radii BC and DE attenuate algebraically in time. For the radius BC (and similarly for DE) $\xi = r\exp(i\varpi\pi)$, and the integral can be evaluated at large t through Watson's lemma, as the integrand is expandable in a series of rational powers ($\mathscr{I}(q,\xi)$ is replaced by $\mathfrak{I}(q,r)$):

$$\lim_{t\to\infty}\int_0^{\infty} \mathfrak{I}(q,r)e^{-rt}\, dr \sim a\Gamma\left(\frac{\alpha}{\mathscr{R}}\right) t^{-\alpha/\mathscr{R}}, \quad (3.15)$$

with $\mathscr{R}$ being the least common denominator of all fractional powers in I, α a positive integer, whose exact value depends on that power series and is unimportant, and a a constant. The above argument holds as long as the radius is located in the left open half plane, i.e., $1/2 < \varpi < 1$ for both species. Otherwise the inverse Laplace transform does not exist.

To exemplify, in the simpler case $\epsilon_j = \delta_j - \gamma_j > 0$, $j \in \{1, 2\}$, $\varpi = \min\{\delta_1, \delta_2\}$, this limitation means that δ_j cannot drop below the value of 1/2. In a situation with $\delta_j = \gamma_j$ ($\epsilon_j = 0$) or $\delta_j = 0$ ($\epsilon_j < 0$), the limitation on the value of $\varpi = \gamma_j$ leads to $1/2 < \gamma < 1$.

Limiting the value of the anomaly exponents so that $1/2 < \varpi < 1$, the inverse Laplace transform equals to the residue integral. To evaluate the latter, introduce $\sigma = \xi^{1/\mathscr{R}}$, with $\mathscr{R}$ being the least common denominator of all fractional powers in I. Thus I will become a ratio of two polynomials, which upon decomposition into a sum of rational functions and Laplace transform inversion will yield exponentially growing terms if at least one of the poles is located in the open right half plane, i.e., if

$$\arg\sigma \in (-\pi/2, \pi/2). \quad (3.16)$$

Now combining the two successive power transformations ϖ and $1/\mathscr{R}$, the contour will change accordingly (solid curve in Figure 1) and the instability sector will become $(-\varpi\pi/(2\mathscr{R}), \varpi\pi/(2\mathscr{R}))$. The generalised instability criterion then becomes, in terms of the dispersion relation roots s_*,

$$\arg s_*^{\varpi/\mathscr{R}} \in (-\varpi\pi/(2\mathscr{R}), \varpi\pi/(2\mathscr{R})). \quad (3.17)$$

4 Dispersion relations

The following cases are considered:

$$\begin{array}{ll} (a) & \gamma_1 = \gamma_2 = \delta_1 = \delta_2 \\ (b) & \gamma_1 = \gamma_2 = \gamma, \quad \delta_j = 0, \quad j \in \{1,2\} \\ (c) & \gamma_1 = \gamma_2 = \gamma, \quad \delta_j = 1, \quad j \in \{1,2\} \\ (d) & \gamma_1 = \gamma_2 = \gamma, \quad \delta_1 = \delta_2 = \delta, \quad \gamma \neq \delta \\ (e) & \gamma_1 \neq \gamma_2, \quad \delta_1 \neq \delta_2, \quad \gamma_j \neq \delta_j, \quad j \in \{1,2\}. \end{array} \tag{4.1}$$

The respective dispersion relations are (the asterisk in the designation of the roots s_* is omitted for convenience):

$$s^{2\gamma} + ((1+d)q^2 - \mathrm{tr}\nabla\mathbf{f})s^{\gamma} + d\,q^4 - \mathrm{tr}_w\nabla\mathbf{f}\,q^2 + \det\nabla\mathbf{f} = 0, \tag{4.2a}$$

$$d\,q^4 s^{-2\gamma} + (1 + d - \mathrm{tr}_w\nabla\mathbf{f})q^2 s^{-\gamma} + \det\nabla\mathbf{f} - \mathrm{tr}\nabla\mathbf{f} + 1 = 0, \tag{4.2b}$$

$$s^2 + (1+d)q^2 s^{2-\gamma} - \mathrm{tr}\nabla\mathbf{f}\,s + d\,q^4 s^{2(1-\gamma)} - \mathrm{tr}_w\nabla\mathbf{f}\,q^2 s^{1-\gamma} + \det\nabla\mathbf{f} = 0, \tag{4.2c}$$

$$s^{2\delta} + (1+d)q^2 s^{2\delta-\gamma} - \mathrm{tr}\nabla\mathbf{f}\,s^{\delta} + d\,q^4 s^{2(\delta-\gamma)} - q^2\mathrm{tr}_w\nabla\mathbf{f}\,s^{\delta-\gamma} + \det\nabla\mathbf{f} = 0, \tag{4.2d}$$

$$\begin{aligned} & s^{\delta_1+\delta_2} + d\,q^2 s^{\delta_1+\delta_2-\gamma_2} + q^2 s^{\delta_1+\delta_2-\gamma_1} - \nabla f_{22} s^{\delta_1} - \nabla f_{11} s^{\delta_2} \\ & + d\,q^4 s^{\delta_1-\gamma_1+\delta_2-\gamma_2} - \nabla f_{22} q^2 s^{\delta_1-\gamma_1} - d\nabla f_{11} q^2 s^{\delta_2-\gamma_2} + \det\nabla\mathbf{f} = 0, \end{aligned} \tag{4.2e}$$

wherein $\mathrm{tr}\nabla\mathbf{f}$, $\mathrm{tr}_w\nabla\mathbf{f} = d\nabla f_{11} + \nabla f_{22}$ and $\det\nabla\mathbf{f}$ are the trace, weighted trace and determinant of the sensitivity matrix $\nabla\mathbf{f}$.

Case (a). Equation (4.2 a), corresponding to anomalous reaction and diffusion with the same exponent, is a quadratics in s^{γ}, identical to the normal one. Hence the curve $s^{\gamma}(q^2)$ has a bell shape and a range of unstable wave numbers $q_-^2 < q^2 < q_+^2$, again identical to that obtained for normal diffusion. If $s^{\gamma} > 0$, then there is a real root $s > 0$, which is unstable. If $s^{\gamma} < 0$, no instability ensues, since $\varpi = \gamma$. So all oscillatory modes are stable (supported by numerical simulations in [11]). This is a particular extension of the normal model with $\epsilon_j = 0$, $j \in \{1,2\}$, drawing a borderline between more general models with $\epsilon_j \gtrless 0$ treated below.

Case (b). Equation (4.2b) has a solution in closed form

$$2d\,q^2 s^{-\gamma} = -(1 + d - \mathrm{tr}_w\nabla\mathbf{f}) \pm \sqrt{\Delta_{(d)}}, \tag{4.3}$$

$$\Delta_{(d)} = (1 + d - \mathrm{tr}_w\nabla\mathbf{f})^2 - 4(\det\nabla\mathbf{f} - \mathrm{tr}\nabla\mathbf{f} + 1).$$

The discriminant may be of either sign. When $\Delta_{(d)} < 0$, θ_Δ denotes the argument of s (with the proper caution concerning the arctangent branches):

$$\theta_\Delta = \pm\frac{1}{\gamma}\arctan\frac{\sqrt{|\Delta_{(d)}|}}{1 + d - \mathrm{tr}_w\nabla\mathbf{f}}. \tag{4.4}$$

The relevant parameter is $\varpi = \gamma$, giving the instability sector as $\big(-\gamma\pi/(2\mathscr{R}), \gamma\pi/(2\mathscr{R})\big)$ with $\mathscr{R}$ being the denominator of the reduced fraction γ. Then one arrives to the following

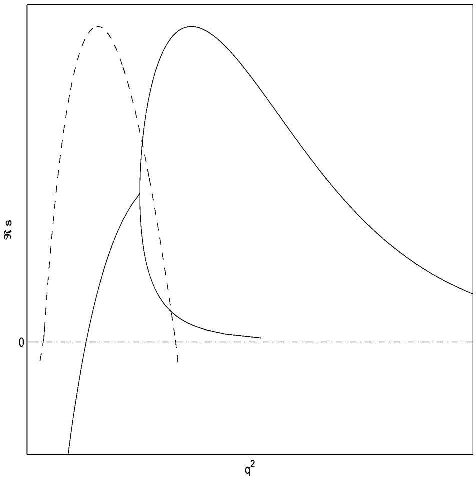

FIGURE 2. Normal ($s(\mathfrak{q}^2)$, dashed) and anomalous ($s(q^2)$, solid) growth rate curves.

possibilities of root locations:

$$\begin{array}{llll} \Delta_{(d)} > 0, & 1 + d - \mathrm{tr}_w \nabla \mathbf{f} > 0, & \arg s = \pi/\gamma > \pi\gamma/(2\mathscr{R}) & \text{stable} \\ \Delta_{(d)} > 0, & 1 + d - \mathrm{tr}_w \nabla \mathbf{f} < 0, & s \in \Re & \text{unstable} \\ \Delta_{(d)} < 0, & & \theta_\Delta \in (-\gamma\pi/(2\mathscr{R}), \gamma\pi/(2\mathscr{R})) & \text{unstable} \\ & & \theta_\Delta \notin (-\gamma\pi/(2\mathscr{R}), \gamma\pi/(2\mathscr{R})) & \text{stable.} \end{array} \tag{4.5}$$

Any of the above cases might be reached upon adjustment of the system parameters. Note that the instability type does not depend on the wave number and is either monotonic or oscillatory.

Case (c). Relation (4.2c) corresponds to the model with anomalous diffusion but normal reaction. Below is a brief analysis of its major properties, pertinent mostly as a limiting case of the more general model (*d*). For a detailed derivation see [15]. The essential difference between equation (4.2c) and its normal counter-part is the appearance of short wave ($q \to \infty$) instability. Designate $\mathfrak{q}^2 \overset{def}{=} q^2 s^{1-\gamma}$ and note that (4.2c) takes the form of a normal relation where q is replaced by $\mathfrak{q}$. Hence in the plane $(\mathfrak{q}^2, s)$ the curve $s(\mathfrak{q}^2)$ is bell-shaped (identically to normal diffusion). As an immediate conclusion follows that if d is large enough, there exists a range of unstable $\mathfrak{q}$, i.e., $\{\mathfrak{q} \mid s(\mathfrak{q}^2) > 0\}$. At the range end points $\{\mathfrak{q} \mid s(\mathfrak{q}^2) = 0\}$ the quantity $\mathfrak{q}^2$ remains finite, whereas s vanishes. The only way to have $s \to 0$ and $q^2 s^{1-\gamma} \nrightarrow 0$ is $q \to \infty$ at the same rate as $s^{1-\gamma} \to 0$. Hence on the plane (q^2, s) at the abscissa points $q^2_\pm$ the curve $s(q^2)$ approaches zero as q tends to infinity, forming two real branches of decaying magnitude. Outside the range of unstable $\mathfrak{q}$ the growth rate s is complex. Figure 2 depicts a typical example. It is seen that the neutral curve changes essentially when the anomaly exponent γ drops below unity, rendering the infinitely short wave numbers unstable. This effect is analysed in more detail below for the more general case of anomalous reaction.

Case (d). Analysis of (4.2d) focuses on the behaviour of $s(q)$ in the limit $q \gg 1$. In this limit the roots belong to one of two groups: roots of decaying magnitude, i.e., $\lim_{q\to\infty}|s| = 0$ (later on 'decaying roots'), or roots of diverging magnitude, i.e., $\lim_{q\to\infty}|s| = \infty$ (later on 'diverging roots'). First, seek decaying roots of (4.2d) in the form

$$s \sim q^{\mu_1} w_1 \left(1 + \sum_{j=2}^{\infty} q^{\mu_j} w_j\right), \quad \mu_j < 0,\ w_j \sim O(1)\ \forall\ j, \quad \mu_{j+1} < \mu_j\ \forall\ j \geqslant 2. \tag{4.6}$$

Comparison of the powers of q yields

$$\mu_1 = -\frac{2}{\delta - \gamma}. \tag{4.7}$$

Thus a decaying solution exists only for $\delta > \gamma$. Solving $O(q^0)$ equation

$$d\, w_1^{2(\delta-\gamma)} - \mathrm{tr}_w \nabla \mathbf{f}\, w_1^{\delta-\gamma} + \det \nabla \mathbf{f} = 0, \tag{4.8}$$

$$w_1^{\delta-\gamma} = \frac{1}{2d}\left(\mathrm{tr}_w \nabla \mathbf{f} \pm \sqrt{\mathrm{tr}_w^2 \nabla \mathbf{f} - 4d \det \nabla \mathbf{f}}\right).$$

When the discriminant is positive, both roots are real and positive, entailing monotonically unstable modes. The vanishing of the discriminant

$$\mathrm{tr}_w^2 \nabla \mathbf{f} - 4d \det \nabla \mathbf{f} = 0 \tag{4.9}$$

defines a critical coefficient d_M, where monotonic instability disappears. There are two options: to view the entries of the matrix $\nabla \mathbf{f}$ as fixed and then $\mathrm{tr}_w \nabla \mathbf{f}$ varies with d, or a rather peculiar approach to render the parameter $\mathrm{tr}_w \nabla \mathbf{f}$ fixed. With the first approach (4.9) is a quadratic equation for d and its solution (only one root d_M satisfies $d_M > 1$) defines the monotonic instability domain $d > d_M$. Conversely, taking $\mathrm{tr}_w \nabla \mathbf{f}$ fixed, an explicit expression

$$d_M = \frac{\mathrm{tr}_w^2 \nabla \mathbf{f}}{4 \det \nabla \mathbf{f}} \tag{4.10}$$

is obtained, but the corresponding instability domain becomes $d < d_M$. The latter approach simplifies greatly the derivation of the oscillatory instability threshold, and is adopted below.

With the special values $\gamma = \delta = 1$ the normal monotonic modes are recovered. By (4.8) the threshold of monotonic instability remains unchanged with the inclusion of anomaly. However, whilst a normal system turns stable above this threshold, in an anomalous one a negative discriminant introduces oscillatory modes, unstable as long as s^{ϖ} is located within the instability sector. Define the oscillatory critical coefficient d_O as the threshold of absolute stability, i.e., the point where the argument of s exceeds the sector boundary. To leading order,

$$\arg w_1^{\delta} \sim \frac{\delta}{\delta - \gamma} \arctan \sqrt{\frac{d}{d_M} - 1} = \frac{\delta \pi}{2}, \tag{4.11}$$

giving

$$d_O \sim d_M \cos^{-2}\left(\frac{\pi}{2}(\delta - \gamma)\right). \tag{4.12}$$

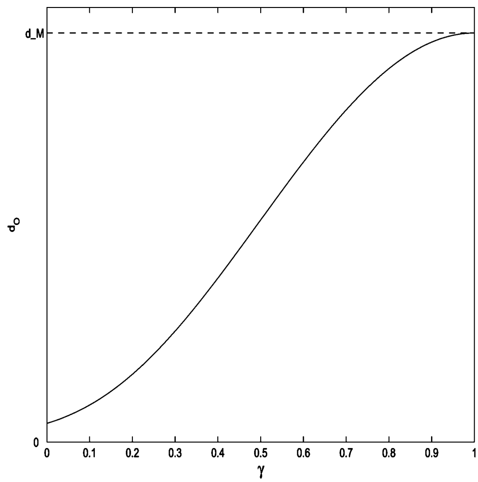

FIGURE 3. Stability threshold diffusion coefficients' ratio d_{O} (oscillatory, solid) and d_{M} (monotonic, dashed) versus the anomaly exponent γ in a system with normal reaction ($\delta = 1$).

The first correction is found via $O(q^{-2\delta/(\delta-\gamma)})$ equation:

$$\mu_2 = -\frac{2\delta}{\delta-\gamma}, \quad w_2 = \mp\frac{(1+d)w_1^{\delta} - \mathrm{tr}\nabla\mathbf{f}\, w_1^{\gamma}}{(\delta-\gamma)\mathrm{tr}_w\nabla\mathbf{f}\sqrt{1-d/d_{\mathrm{M}}}}\,. \tag{4.13}$$

The contribution of $\arg w_2$ is negligible when $\tan\arg w_1$ diverges at d_O. Since all higher orders will have the same property,

$$d_{\mathrm{O}} = d_{\mathrm{M}}\cos^{-2}\left(\frac{\pi}{2}(\delta-\gamma)\right). \tag{4.14}$$

Similarly to (4.10), this may be regarded as a quadratic equation in d for fixed entries of $\nabla\mathbf{f}$,

$$d_{\mathrm{O}} = \frac{\mathrm{tr}_w^2\nabla\mathbf{f}(d_{\mathrm{O}})}{4\det\nabla\mathbf{f}}\cos^{-2}\left(\frac{\pi}{2}(\delta-\gamma)\right), \tag{4.15}$$

and then the oscillatory instability domain is $d_{\mathrm{O}} < d < d_{\mathrm{M}}$, or as an explicit expression if $\mathrm{tr}_w\nabla\mathbf{f}$ is fixed and then the corresponding domain is $d_{\mathrm{M}} < d < d_{\mathrm{O}}$.

For a system with anomalous diffusion ($\gamma < 1$), but normal reaction ($\delta = 1$), equation (4.14) simplifies to

$$d_{\mathrm{O}} = d_{\mathrm{M}}\sin^{-2}\left(\frac{\pi\gamma}{2}\right), \tag{4.16}$$

coinciding with the result obtained in [15]. To clarify this result in the case of the usual approach with $\nabla\mathbf{f}$ fixed, the solution of (4.14) is plotted as a function of the anomaly exponent in Figure 3. Note the instability threshold decrease as the system grows more anomalous (smaller values of γ).

A typical example of the way the growth rate function changes with d is shown in Figure 4. For a normal sub-critical value of d two real branches correspond to monotonic unstable modes. The complex branch located within the instability sector corresponds to oscillatory unstable modes. With an anomalous sub-critical value of d the real branches

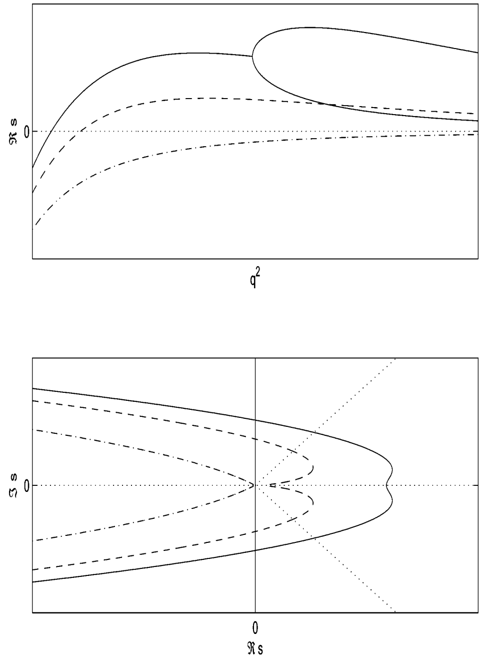

FIGURE 4. Growth rate function versus wave number (upper) and its polar map (lower) for $\gamma = 0.5, \delta = 0.9$ and three values of d: normal sub-critical (solid, one complex and two real unstable branches), anomalous sub-critical (dashed, one complex unstable branch) and anomalous super-critical (dash-dotted, one complex stable branch). The instability sector is shown by a dotted line. All complex roots are shown with their conjugate counterparts.

disappear, but the complex branch still lies within the instability sector. For an anomalous super-critical d the remaining complex branch is stable.

It is possible to obtain a diverging solution, i.e., a branch with $|s(q)| \to \infty$ as $q \to \infty$. Using the expansion

$$s \sim q^{v_1} w_1 \left(1 + \sum_{j=2}^{\infty} q^{v_j} w_j \right), \quad v_1 > 0, \; w_j \sim O(1) \; \forall \; j, \quad v_{j+1} < v_j < 0 \; \forall \; j \geqslant 2 \tag{4.17}$$

and comparing the powers of q in (4.2d),

$$v_1 = \frac{4\delta}{\gamma}. \tag{4.18}$$

Of course, a solution of this nature is valid for a non-vanishing δ (otherwise by (4.17) it does not diverge in magnitude). In particular, the solution of case (b) cannot be obtained from (4.17) by a simple substitution of $\delta = 0$.

Solving the $O(q^{4\delta/\gamma})$ equation,

$$d w_1^{-2\gamma} + (1+d) w_1^{-\gamma} + 1 = 0, \qquad w_{11}^{\gamma} = -1, \quad w_{12}^{\gamma} = -d, \tag{4.19}$$

where w_{jk} is the k-th branch of the j-th term in expansion (4.17). So far there was no restriction on the relation between δ and γ. The first correction is found via $O(q^{2\delta/\gamma})$ equation:

$$v_2 = -\frac{2\delta}{\gamma}, \tag{4.20}$$

$$w_{21} = -\frac{w_{11}^{-\delta}}{\gamma(1+d)}(d-1)\nabla f_{11}, \quad w_{22} = -\frac{w_{12}^{-\delta}}{\gamma(1+d)}(1-d)\nabla f_{22}.$$

Bearing in mind the constraint $d > 1$ and the signs of the entries of $\nabla\mathbf{f}$ [14], w_1^δ will determine the sign of the argument of w_2:

$$\arg w_2 = \pi\frac{\delta}{\gamma}. \tag{4.21}$$

Hence the sign of w_2 will be determined by the relation between δ and γ. There seems to be no further importance to that relation for the diverging solution.

As to stability properties, $\pi\delta/(\mathscr{R}\gamma) > \pi\delta/(2\mathscr{R})$ ($\mathscr{R}$ is an integer and $\gamma < 1$), so that diverging solutions exhibit no instability at the short wave limit.

All results obtained for case (d) generalise the derivation for the case of equal anomaly exponents γ with $\delta = 1$ [15]. Thus for $\gamma < \delta$ there are both decaying and diverging solutions, similarly to $\delta = 1$ case (there $\delta = \delta_{\max} = 1$, and the results are valid for the widest possible range of γ). Conversely, for $0 < \delta < \gamma$ there is just the diverging solution, which is stable (the possibility of instability at moderate wave numbers is not excluded). The reason for such asymmetry about γ, δ is as follows.

Similarly to the $\delta = 1$ case, denote

$$\mathfrak{q}^2 = q^2 s^{\delta-\gamma}. \tag{4.22}$$

Then (4.2d) reads

$$s^{2\delta} + s^\delta[(1+d)\mathfrak{q}^2 - \mathrm{tr}\nabla\mathbf{f}] + d\,\mathfrak{q}^4 - \mathfrak{q}^2\mathrm{tr}_w\nabla\mathbf{f} + \det\nabla\mathbf{f} = 0, \tag{4.23}$$

which gives the bell-shaped curve $s^\delta(q^2 s^{\delta-\gamma})$. This argument holds also for case (a), so that $\delta \geqslant \gamma$. When $s \to 0$ and $\mathfrak{q}^2 \to q_\pm^2$, the two tails of the decaying solutions with $q \to \infty$ are obtained. This clarifies the appearance of the relation $\gamma < \delta$ – without this condition it is impossible to have s infinitesimal and keep $q^2 s^{\delta-\gamma}$ finite. If $0 < \delta < \gamma$, the bell-shaped curve is irrelevant, and only the diverging solution exists. The solution in the limiting case $\delta = 0$ also diverges in magnitude for $q \gg 1$, yet is of a different nature and has been treated separately in case (b). The limit $\delta = \gamma$, on the other hand, is worth further insight.

The case $\delta = \gamma$ is singular in the following sense. Denote $\epsilon = \delta - \gamma$. For an infinitesimal, but non-vanishing, value of ϵ the function s^ϵ drops from unity to zero at $s = 0$ over an infinitesimally narrow range of s. The bell-shaped function $s^\gamma(q^2)$ for $\epsilon = 0$ becomes $s^{\gamma+\epsilon}(q^2 s^\epsilon)$, replacing the finite intersection points with the abscissa by two decaying tails. Nevertheless, the transition from $\epsilon = 0$ to $\epsilon > 0$ is smooth in the sense that the bell shape changes very little. Figure 5 shows the curves for $\delta - \gamma = 0$ and $\delta - \gamma \ll 1$. In the latter case the curve is close to normal, but possesses two short wave tails.

Case (e). The most general model in this context is (e). Again, the analysis focuses on the growth rate dependence on the wave number at the limit $q \gg 1$. Just like in (b), the sign

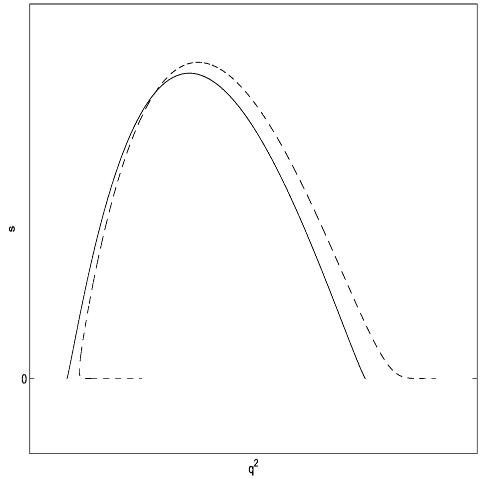

FIGURE 5. Transition from $\delta = \gamma$ (solid) to $\delta = \gamma + \epsilon$ (dashed) for $\gamma = 0.9, \epsilon = 0.01$. For the latter, $s \in \mathbb{C}$ in the range $q < q_-$.

of $\epsilon_j = \delta_j - \gamma_j$, $j \in \{1, 2\}$, determines the solution character. Hereby the solutions are presented to leading order only. All higher order corrections appear in the appendix.

First, consider a system with $\epsilon_j > 0$, $j \in \{1, 2\}$. One may expect that decaying solutions exist. By expansion (4.6), comparison of the powers of q in (4.2e) reveals two branches for each of the cases $\epsilon_1 \gtrless \epsilon_2$, generalising $\gamma_1 \lessgtr \gamma_2$ with $\delta_j = 1$, $j \in \{1, 2\}$ [15]. For $\epsilon_1 > \epsilon_2$ the leading terms are given by equations of $O(q^0)$

$$\mu_{11} = -\frac{2}{\epsilon_2}, \quad w_{11}^{\epsilon_2} = \frac{\det \nabla \mathbf{f}}{d \nabla f_{11}}, \tag{4.24}$$

and $O(q^{2(1-\epsilon_2/\epsilon_1)})$

$$\mu_{12} = -\frac{2}{\epsilon_1}, \quad w_{12}^{\epsilon_1} = \nabla f_{11}. \tag{4.25}$$

For $\epsilon_1 < \epsilon_2$ the leading terms are given by equations of $O(q^0)$

$$\mu_{11} = -\frac{2}{\epsilon_1}, \quad w_{11}^{\epsilon_1} = \frac{\det \nabla \mathbf{f}}{\nabla f_{22}}, \tag{4.26}$$

and $O(q^{2(1-\epsilon_1/\epsilon_2)})$

$$\mu_{12} = -\frac{2}{\epsilon_2}, \quad w_{12}^{\epsilon_2} = \frac{\nabla f_{22}}{d}. \tag{4.27}$$

The functions w_{1j} are an immediate generalisation of the case $\delta_j = 1$, $j \in \{1, 2\}$ [15]. As to stability characteristics, if $\epsilon_1 > \epsilon_2$, both branches are real positive and thus unstable. Conversely, if $\epsilon_1 < \epsilon_2$, both branches are complex with arguments lying outside the instability sector:

$$\arg w_{1j}^{\varpi/\mathscr{R}} = \varpi\pi/(\epsilon_j \mathscr{R}) > \varpi\pi/(2\mathscr{R}), \quad j \in \{1, 2\}. \tag{4.28}$$

Now suppose $\epsilon_j < 0$, $j \in \{1,2\}$. With these relations between the anomaly exponents it is impossible to obtain a consistent expansion of form (4.6), and hence no decaying solutions exist. This property makes the relations $\epsilon_j \gtrless 0$ rather interesting. First, suppose $\epsilon_1 < 0, \epsilon_2 > 0$. By (4.6), a decaying root of (4.2e) ensues at order $O(q^{2(1-\epsilon_1/\epsilon_2)})$:

$$\mu_1 = -\frac{2}{\epsilon_2}, \quad w_1^{\epsilon_2} = \nabla f_{22}/d, \tag{4.29}$$

coincident with the leading order of one of the solutions obtained above with $\epsilon_j > 0$, $j \in \{1,2\}$. Conversely, suppose $\epsilon_2 < 0, \epsilon_1 > 0$. A decaying root ensues at order $O(q^{2(1-\epsilon_2/\epsilon_1)})$:

$$\mu_1 = -\frac{2}{\epsilon_1}, \quad w_1^{\epsilon_1} = \nabla f_{11}, \tag{4.30}$$

again coincident with the leading order of one of the solutions obtained above with $\epsilon_j > 0$, $j \in \{1,2\}$. Thus, the number of positive ϵ_j determines the number of decaying roots. To leading order the solution of a single $\epsilon_j < 0$ coincides with one of the solutions for both $\epsilon_j > 0$, and the stability characteristics follow, since the sign of ϵ_j does not change the instability sector.

For diverging solutions the sign of ϵ_j is unimportant, and the relevant distinction is $\gamma_1 \lessgtr \gamma_2$. For $\gamma_1 < \gamma_2$ the two branches ensue by equations of order $O(q^{2(\delta_1+\delta_2)/\gamma_1})$

$$v_{11} = \frac{2}{\gamma_1}, \quad w_{11}^{\gamma_1} = -1, \tag{4.31}$$

and $O(q^{2+2(\delta_2+\epsilon_1)/\gamma_2})$

$$v_{12} = \frac{2}{\gamma_2}, \quad w_{12}^{\gamma_2} = -d. \tag{4.32}$$

For $\gamma_1 > \gamma_2$ the two branches ensue by equations of order $O(q^{2(\delta_1+\delta_2)/\gamma_2})$

$$v_{11} = \frac{2}{\gamma_2}, \quad w_{11}^{\gamma_2} = -d, \tag{4.33}$$

and $O(q^{2+2(\delta_1+\epsilon_2)/\gamma_1})$

$$v_{12} = \frac{2}{\gamma_1}, \quad w_{12}^{\gamma_1} = -1. \tag{4.34}$$

To leading order these diverging solutions coincide with the ones obtained in [15] for $\delta_j = 1$, $j \in \{1,2\}$ and $\gamma_1 \neq \gamma_2$, and thus the stability characteristics follow, as the instability sector might only grow narrower with $\delta_j < 1$. The expressions for the corrections generalise the results in the case $\delta_j = 1$. It is also interesting that, to leading order only, these are the solutions obtained above for case (*d*).

Case (*e*), i.e., the situation where one species' rate of reaction is faster than its diffusion, whereas the other species' behaviour is just opposite, enables the general conclusion on the source of short wave monotonic instability: the relation $0 < \epsilon_1 > \epsilon_2$ corresponds to a larger reaction–diffusion scale difference for the activator and suffices for the monotonic instability to persist over a semi-infinite range of wave numbers q.

Table 1 summarises the number of branches, their type and stability properties for the different combinations of anomaly exponents.

Table 1. *Root types and stability properties for different combinations of anomaly exponents (with fixed entries of* $\nabla\mathbf{f}$*)*

Anomaly exponents	Branches		Instability	
$\delta_1 = \delta_2 = \gamma$ $\gamma_1 = \gamma_2 = \gamma$	Bell-shaped 1 real	2 complex tails or all complex	Monotonic $q_- < q < q_+$ $d > d_{\mathrm{M}}$	Oscillatory any q none
$\delta_1 = \delta_2 = \delta$ $\gamma_1 = \gamma_2 = \gamma$ $\delta \neq \gamma$	Decaying 2 real or 1 complex	Diverging 2 complex	Monotonic $q > q_{\min}$ $d > d_{\mathrm{M}}$	Oscillatory $q \gg 1$ $d_{\mathrm{O}} < d < d_{\mathrm{M}}$
$\epsilon_1 = \delta_1 - \gamma_1 > 0$ $\epsilon_2 = \delta_2 - \gamma_2 > 0$ $\epsilon_1 \neq \epsilon_2$	Decaying 2 real or 2 complex	Diverging 2 complex	Monotonic $q \gg 1$ $\epsilon_1 > \epsilon_2$	Oscillatory $q \gg 1$ none
$\epsilon_1 > 0$ $\epsilon_2 < 0$	Decaying 1 real	Diverging 2 complex	Monotonic $q \gg 1,\ \forall\, d$	Oscillatory $q \gg 1$, none
$\epsilon_1 < 0$ $\epsilon_2 > 0$	Decaying 1 complex	Diverging 2 complex	Monotonic any q, none	Oscillatory $q \gg 1$, none
$\epsilon_1 < 0$ $\epsilon_2 < 0$	Decaying none	Diverging 2 complex	Monotonic any q, none	Oscillatory any q, none

Notation $q_\pm$ stands for end points of the normal unstable interval; $q_{\min}$ is system-dependent and finite.

5 Concluding remarks

The investigation has treated a two-species anomalous reaction–anomalous diffusion system. Generalised Turing instability condition revealed an anomaly-dependent restriction, more severe than the analogue for normal reaction–anomalous diffusion. Here instability ensues if the dispersion relation roots s satisfy

$$\arg s^{\varpi/\mathscr{R}} \in \left(-\frac{\varpi\pi}{2\mathscr{R}}, \frac{\varpi\pi}{2\mathscr{R}}\right),$$

with $\mathscr{R}$ being the least common denominator of all fractional powers.

In the special case of 'maximally anomalous' reaction–anomalous diffusion ($\gamma_1 = \gamma_2 = \gamma,\ \ \delta_1 = \delta_2 = 0$) the system is unstable for a certain combination of parameters.

In the case of all equal exponents anomalous reaction–anomalous diffusion ($\gamma_1 = \delta_1 = \gamma_2 = \delta_2 = \gamma$) instability mechanism is akin to normal, with the same dependence on the coefficients' ratio d. The curve $s^\gamma(q)$ is of a bell shape. When $d > d_M$, the maximal growth rate is positive and a range of unstable wave numbers $q_- < q < q_+$ exists. Outside that range $s^\gamma < 0$ and hence there are two complex tails. When $d < d_M$ $s^\gamma < 0$ for all q, i.e., the growth rate is always complex. No oscillatory instability is observed.

The case of anomalous reaction–anomalous diffusion with equal exponents for both species ($\gamma_1 = \gamma_2 = \gamma,\ \ \delta_1 = \delta_2 = \delta$) is a generalisation of the normal reaction–anomalous diffusion. The system is monotonically unstable when the coefficients' ratio d is above the normal critical value d_M, oscillatorily unstable at the range $d_O < d < d_M$ and stable below d_O, the anomalous critical value.

In the general case of anomalous reaction–anomalous diffusion with distinct exponents ($\gamma_1 \neq \gamma_2$, $\delta_1 \neq \delta_2$, $\gamma_j \neq \delta_j$, $j \in \{1,2\}$) the number of branches decaying at the short wave limit is determined by the number of positive ϵ_j, i.e., the number of species whose diffusion is more anomalous than reaction. The decay in the growth rate amplitude is a necessary, yet insufficient, condition for instability, ensuing if and only if both $\epsilon_1 > \epsilon_2$ and $\epsilon_1 > 0$ hold (ϵ_2 might be negative). Relation $\epsilon_1 > \epsilon_2$ generalises the constraint of faster diffusion for the inhibitor as an essential condition for instability onset (modelled in normal reaction–diffusion via a larger diffusion coefficient) and presents the boundary for appearance of monotonic instability within the framework of this model.

Whenever there is a positive real branch of $s(q)$, oscillatory instability is possible at a finite interval of wave numbers. The reason is that without diffusion ($q = 0$) the system was taken stable, whereas monotonic instability ($s > 0$) was shown to occur at $q \gg 1$. Oscillatory unstable modes provide a smooth transition from the absolute stability at long wave disturbances to monotonic instability at short waves. A typical system that exhibits this kind of transition is the limiting case of anomalous diffusion–anomalous reaction with equal exponents for both species $\gamma_1 = \gamma_2 = \gamma$, $\delta_1 = \delta_2 = \delta$ ($\epsilon_1 = \epsilon_2 = \delta - \gamma$). A system with $\epsilon_1 > \epsilon_2$, $\epsilon_1 > 0$ possesses a similar property with the only difference that the real branches exist regardless of the value of the diffusion coefficients' ratio d, so that the oscillatory instability is always accompanied by the monotonic one.

Diverging branches conform to oscillatory modes of a different nature. They exist for all combinations of anomaly exponents, are always stable and do not alter the overall system stability properties.

Thus, sole existence of monotonic modes is characteristic to systems with equal anomaly exponents for reaction and diffusion processes. A normal reaction–diffusion system is just a limit case of a continuum of such systems. Arbitrary combinations of anomaly exponents lead to appearance of oscillatory modes, at least in a bounded interval of wave numbers. Monotonic modes exist whenever the inhibitor diffusion is faster than that of the activator. From that standpoint, a normal reaction–diffusion system is again a limit case, where enhanced inhibitor diffusion is obtained by a larger diffusion coefficient. The main innovation of the current work was establishing a smooth transient between stability properties of more general systems.

In conclusion, it should be noted that while Turing patterns have been obtained experimentally, e.g., in gel reactors [22], they have not yet been observed in systems with anomalous diffusion. Bearing in mind the experimental evidence of anomalous diffusion of molecules in gels [9], one can expect experiments in gel solvents to reveal Turing instability in sub-diffusive systems. Such experiments could verify model (2.1) and measure the values of the parameters γ_j,δ_j used in the model. Precise measures of anomaly exponents for molecules diffusing and reacting within living cells also remains an unmet challenge.

Non-linear theory and more complicated memory models are of interest for future research. As a first step, weakly non-linear dynamics might be tackled, for instance amplitude equations ought to be derived. Strongly non-linear analysis will definitely require development of special numerical techniques to deal with the singular nature of the fractional derivative.

Appendix

Detailed root analysis for case (4.2e)

Substituting (4.6) into (4.2e), for $\epsilon_1 > \epsilon_2$ the leading terms are given by equations of $O(q^0)$

$$\mu_{11} = -\frac{2}{\epsilon_2}, \quad w_{11}^{\epsilon_2} = \frac{\det \nabla \mathbf{f}}{d \nabla f_{11}}, \tag{A1}$$

and $O(q^{2(1-\epsilon_2/\epsilon_1)})$

$$\mu_{12} = -\frac{2}{\epsilon_1}, \quad w_{12}^{\epsilon_1} = \nabla f_{11}. \tag{A2}$$

Corrections are slightly more complicated in this case. For the first branch, if $\epsilon_1 - \epsilon_2 < \delta_2$, the next balanced order is $O(q^{2(1-\epsilon_1/\epsilon_2)})$:

$$\mu_{21} = 2\left(1 - \frac{\epsilon_1}{\epsilon_2}\right), \quad w_{21} = \frac{(\det \nabla \mathbf{f} - \nabla f_{11} \nabla f_{22}) w_{11}^{\epsilon_1}}{\epsilon_2 \nabla f_{11} \det \nabla \mathbf{f}}. \tag{A3}$$

If, however, $\epsilon_1 - \epsilon_2 > \delta_2$, the next balanced order becomes $O(q^{-2\delta_2/\epsilon_2})$:

$$\mu_{21} = -\frac{2\delta_2}{\epsilon_2}, \quad w_{21} = -\frac{w_{11}^{\gamma_2}}{d\epsilon_2}. \tag{A4}$$

Obviously, the latter possibility is not feasible with $\delta_2 = 1$. For the second branch the correction is given by $O(q^0)$ equation:

$$\mu_{22} = -2\left(1 - \frac{\epsilon_2}{\epsilon_1}\right), \quad w_{22} = \frac{\nabla f_{11} \nabla f_{22} - \det \nabla \mathbf{f}}{d\epsilon_1 \nabla f_{11} w_{12}^{\epsilon_2}}. \tag{A5}$$

For $\epsilon_1 < \epsilon_2$ the leading terms are given by equations of $O(q^0)$,

$$\mu_{11} = -\frac{2}{\epsilon_1}, \quad w_{11}^{\epsilon_1} = \frac{\det \nabla \mathbf{f}}{\nabla f_{22}}, \tag{A6}$$

and $O(q^{2(1-\epsilon_1/\epsilon_2)})$

$$\mu_{12} = -\frac{2}{\epsilon_2}, \quad w_{12}^{\epsilon_2} = \frac{\nabla f_{22}}{d}. \tag{A7}$$

Similarly, for the first branch, if $\epsilon_2 - \epsilon_1 < \delta_1$, the next balanced order is $O(q^{2(1-\epsilon_2/\epsilon_1)})$:

$$\mu_{21} = 2\left(1 - \frac{\epsilon_2}{\epsilon_1}\right), \quad w_{21} = \frac{(\det \nabla \mathbf{f} - \nabla f_{11} \nabla f_{22}) d\, w_{11}^{\epsilon_2}}{\epsilon_1 \nabla f_{22} \det \nabla \mathbf{f}}, \tag{A8}$$

whereas if $\epsilon_2 - \epsilon_1 > \delta_1$, the next balanced order becomes $O(q^{-2\delta_1/\epsilon_1})$:

$$\mu_{21} = -\frac{2\delta_1}{\epsilon_1}, \quad w_{21} = -\frac{w_{11}^{\gamma_1}}{\epsilon_1}. \tag{A9}$$

Again, the latter possibility is not feasible when $\delta_1 = 1$. For the second branch the correction is given by $O(q^0)$ equation:

$$\mu_{22} = -2\left(1 - \frac{\epsilon_1}{\epsilon_2}\right), \quad w_{22} = \frac{\nabla f_{11} \nabla f_{22} - \det \nabla \mathbf{f}}{\epsilon_2 \nabla f_{22} w_{12}^{\epsilon_1}}. \tag{A10}$$

For all combinations the functions w_{1j} are an immediate generalisation of the case $\delta_j = 1$, $j \in \{1,2\}$ [15]. As to stability characteristics, if $\epsilon_1 > \epsilon_2$, both branches are real positive and thus unstable. Conversely, if $\epsilon_1 < \epsilon_2$, both branches are complex with arguments lying outside the instability sector:

$$\arg w_{1j}^{\varpi/\mathscr{R}} = \varpi\pi/(\epsilon_j\mathscr{R}) > \varpi\pi/(2\mathscr{R}), \quad j \in \{1,2\}. \tag{A11}$$

With $\epsilon_j < 0$, $j \in \{1,2\}$, no decaying solutions exist. To solve for only one negative exponent, first, suppose $\epsilon_1 < 0, \epsilon_2 > 0$. By (4.6), a decaying root of (4.2e) ensues at order $O(q^{2(1-\epsilon_1/\epsilon_2)})$,

$$\mu_1 = -\frac{2}{\epsilon_2}, \quad w_1^{\epsilon_2} = \nabla f_{22}/d, \tag{A12}$$

coincident with the leading order of one of the solutions obtained above with $\epsilon_j < 0$, $j \in \{1,2\}$. However, the correction differs and is distinct for $\epsilon_1 + \gamma_2 \gtrless 0$. If $\epsilon_1 + \gamma_2 < 0$, the next balanced order is $O(q^{-2(\epsilon_1+\gamma_2)/\epsilon_2})$,

$$\mu_2 = -\frac{2\delta_2}{\epsilon_2}, \quad w_2 = -\frac{w_1^{\delta_2}}{\epsilon_2\nabla f_{22}}, \tag{A13}$$

whereas if $\epsilon_1 + \gamma_2 > 0$, the next balanced order is $O(q^0)$:

$$\mu_2 = -2\left(1 - \frac{\epsilon_1}{\epsilon_2}\right), \quad w_2 = \frac{\nabla f_{11}\nabla f_{22} - \det \nabla \mathbf{f}}{\epsilon_2\nabla f_{22} w_1^{\epsilon_1}}. \tag{A14}$$

Conversely, suppose $\epsilon_2 < 0, \epsilon_1 > 0$. A decaying root ensues at order $O(q^{2(1-\epsilon_2/\epsilon_1)})$,

$$\mu_1 = -\frac{2}{\epsilon_1}, \quad w_1^{\epsilon_1} = \nabla f_{11}, \tag{A15}$$

again coincident with the leading order of one of the solutions obtained above with $\epsilon_j < 0$, $j \in \{1,2\}$. If $\epsilon_2 + \gamma_1 < 0$, the next balanced order is $O(q^{-2(\epsilon_2+\gamma_1)/\epsilon_1})$,

$$\mu_2 = -2\delta_1/\epsilon_1, \quad w_2 = -\frac{w_1^{\delta_1}}{\epsilon_1\nabla f_{11}}, \tag{A16}$$

whereas if $\epsilon_2 + \gamma_1 > 0$, the next balanced order is $O(q^0)$:

$$\mu_2 = -2\left(1 - \frac{\epsilon_2}{\epsilon_1}\right), \quad w_2 = \frac{\nabla f_{11}\nabla f_{22} - \det \nabla \mathbf{f}}{\epsilon_1 d\, \nabla f_{11} w_1^{\epsilon_2}}. \tag{A17}$$

For diverging solutions the sign of ϵ_j is unimportant, and the relevant distinction is $\gamma_1 \lesseqgtr \gamma_2$. For $\gamma_1 < \gamma_2$ the two branches ensue by equations of order $O(q^{2(\delta_1+\delta_2)/\gamma_1})$

$$v_{11} = \frac{2}{\gamma_1}, \quad w_{11}^{\gamma_1} = -1, \tag{A18}$$

and $O(q^{2+2(\delta_2+\epsilon_1)/\gamma_2})$

$$v_{12} = \frac{2}{\gamma_2}, \quad w_{12}^{\gamma_2} = -d. \tag{A19}$$

Correction for the first branch is at order $O(q^{2\delta_2/\gamma_1})$, whereas terms of orders $O(q^{2+2(\delta_1+\epsilon_2)/\gamma_1})$ and $O(q^{2\delta_1/\gamma_1})$ cancel:

$$v_{21} = -\frac{2\delta_1}{\gamma_1}, \quad w_{21} = \frac{\nabla f_{11}}{\gamma_1 w_{11}^{\delta_1}}. \tag{A20}$$

For the second branch terms of orders $O(q^{2(\delta_1+\delta_2)/\gamma_2})$ and $O(q^{2\delta_2/\gamma_2})$ cancel, and the correction ensues at order $O(q^{2+2\epsilon_1/\gamma_2})$:

$$v_{22} = -\frac{2\delta_2}{\gamma_2}, \quad w_{22} = \frac{\nabla f_{22}}{\gamma_2 w_{12}^{\delta_2}}. \tag{A21}$$

For $\gamma_1 > \gamma_2$ the two branches ensue by equations of order $O(q^{2(\delta_1+\delta_2)/\gamma_2})$

$$v_{11} = \frac{2}{\gamma_2}, \quad w_{11}^{\gamma_2} = -d, \tag{A22}$$

and $O(q^{2+2(\delta_1+\epsilon_2)/\gamma_1})$

$$v_{12} = \frac{2}{\gamma_1}, \quad w_{12}^{\gamma_1} = -1. \tag{A23}$$

Correction for the first branch is at order $O(q^{2\delta_1/\gamma_2})$, whereas terms of orders $O(q^{2+2(\delta_2+\epsilon_1)/\gamma_2})$ and $O(q^{2\delta_2/\gamma_2})$ cancel:

$$v_{21} = -\frac{2\delta_2}{\gamma_2}, \quad w_{21} = \frac{\nabla f_{22}}{\gamma_2 w_{11}^{\delta_2}}. \tag{A24}$$

For the second branch terms of orders $O(q^{2(\delta_1+\delta_2)/\gamma_1})$ and $O(q^{2\delta_1/\gamma_1})$ cancel, and the correction ensues at order $O(q^{2+2\epsilon_2/\gamma_1})$:

$$v_{22} = -\frac{2\delta_1}{\gamma_1}, \quad w_{22} = \frac{\nabla f_{11}}{\gamma_1 w_{12}^{\delta_1}}. \tag{A25}$$

The expressions for the corrections generalise the results in the case $\delta_j = 1$. Note that similarly the solutions coincide to four orders of magnitude, yet it is impossible to propose identity to any order.

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