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Weakly nonlinear dynamics in reaction–diffusion systems with Lévy flights

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Abstract
Reaction–diffusion equations with a fractional Laplacian are reduced near a long wave Hopf bifurcation. The obtained amplitude equation is shown to be the complex Ginzburg–Landau equation with a fractional Laplacian. Some of the properties of the normal complex Ginzburg–Landau equation are generalized for the fractional analogue. In particular, an analogue of the Kuramoto–Sivashinsky equation is derived.

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1. Introduction

Random processes characterized by Lévy flights have been discovered several decades ago. Since then, similar processes have been observed in numerous natural phenomena: diffusion and advection in fluids [1], in particular turbulent flows [2] and wave turbulence [3], motion of animals [4], balance control in humans [5] and even progress of seismic foci [6]. At the macroscopic limit, Lévy flights are modeled by a fractional Laplacian operator. The general properties of such processes are reviewed in [7]. Often non-linear kinetics, such as (but not only) chemical reactions create an intricate interaction with the diffusion process, especially at an instability threshold [8]–[10]. Thus an equation combining a fractional diffusion operator (a Laplacian fraction $\gamma/2$ with $1<\gamma\leqslant 2$) and nonlinear kinetics is the simplest model to capture the basic effects of such an interaction.

Understanding of pattern emergence and formation in normal reaction–diffusion systems near a Hopf bifurcation point was achieved by means of amplitude and phase diffusion equations [11]–[14]. The present work follows the course of reduction of the fractional reaction–diffusion model near such a bifurcation in order to obtain and study the fractional analogues of complex Ginzburg–Landau (amplitude) and Kuramoto–Sivashinsky (phase diffusion) equations. The obtained results can be relevant to the problem of mixing by disordered flows in the presence of chemical reactions.

2. Reduction near bifurcation point

Consider a two-species fractional reaction–diffusion system

$$\frac{\partial \mathbf{n}}{\partial t}=\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix}\begin{pmatrix}\mathfrak{D}^{\gamma_1}_{|x|}n_1\\ \mathfrak{D}^{\gamma_2}_{|x|}n_2\end{pmatrix}+\mathbf{f}(\mathbf{n}), \tag{1}$$

where the Laplacian fractional counterpart is of a generally distinct order for each species:

$$\mathfrak{D}^{\gamma}_{|x|}\mathbf{n}(x,t)=-\frac{\sec(\pi\gamma/2)}{2\Gamma(2-\gamma)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{\mathbf{n}(\zeta,t)\,\mathrm{d}\zeta}{|x-\zeta|^{\gamma-1}},\quad 1<\gamma<2, \tag{2}$$

and $\mathbf{n}$, $\mathbf{f}$ and d_j are the species concentration vector, kinetics function and diffusion coefficients, correspondingly. Suppose there exists a uniform steady state $\mathbf{n_0}$ satisfying $\mathbf{f}(\mathbf{n}_0)=\mathbf{0}$, in whose close vicinity $\mathbf{f}$ varies according to a sensitivity matrix $(\nabla\mathbf{f})_{jk}=\partial f_j/\partial n_k$, $j,k\in\{1,2\}$. Then Hopf bifurcation occurs when its trace vanishes. Let us split the matrix as

$$\nabla\mathbf{f}=\nabla\mathbf{f}_0+\epsilon^2\begin{pmatrix}0&0\\0&\mu\end{pmatrix},\quad \mathrm{tr}\,\nabla\mathbf{f}_0=0,\qquad \epsilon\ll 1,\quad 0<\mu\sim O(1), \tag{3}$$

rescale

$$\mathbf{n}(x,t)=\mathbf{N}(\xi,t_0,t_2,\ldots;\epsilon),\qquad \xi=\delta x,\quad t_j=\epsilon^j t,\quad j=0,2,\ldots \tag{4}$$

and expand

$$\mathbf{N}\sim\mathbf{n}_0+\sum_{j=1}^{\infty}\delta_j\mathbf{N}_j(\xi,t_0,t_2,\ldots),\qquad \delta_j=\delta_j(\epsilon). \tag{5}$$

For the leading order reduction only the first slow temporal and spatial scales will be used. Substitution of (5) into (1) and scrutiny of the resulting system led to the following choice of δ and δ_j. If $\gamma_1=\gamma_2=\gamma$, the scales are $\delta_j=\epsilon^j$ and $\delta^\gamma=\epsilon^2$, where the latter ensues by

$$\mathfrak{D}^{\gamma}_{|x|}y(x)=\delta^{\gamma}\mathfrak{D}^{\gamma}_{|\xi|}y(\xi/\delta). \tag{6}$$

Then at order $O(\delta_1)$ a system of linear homogeneous equations at the bifurcation point is obtained:

$$\frac{\partial\mathbf{N}_1}{\partial t_0}-\nabla\mathbf{f}_0\mathbf{N}_1=0. \tag{7}$$

Its solution is

$$\mathbf{N}_1=A(\xi,t_2)\,\mathrm{e}^{\lambda t_0}\mathbf{v}_1+\text{c.c.},\qquad \lambda=\mathrm{i}\omega,$$
$$\mathbf{v}_1=\begin{pmatrix}1\\u\end{pmatrix},\qquad u=\frac{\mathrm{i}\omega-\nabla f_{11}}{\nabla f_{12}},\qquad \omega^2=\det\nabla\mathbf{f}_0. \tag{8}$$

At subsequent orders the system is not homogeneous. At order $O(\delta_3)$ secular non-homogeneous terms coerce a solvability condition for A, alias fractional amplitude or Ginzburg–Landau equation:

$$\frac{\partial A}{\partial t_2}=\frac{\mu}{2}A+\left(\frac{d_2+d_1}{2}+\mathrm{i}\frac{d_2-d_1}{2}\frac{\nabla f_{11}}{\omega}\right)\mathfrak{D}^{\gamma}_{|x|}A-s\,A|A|^2, \tag{9}$$

where s is constant and depends on ω and derivatives of $\mathbf{f}$ up to third order at the bifurcation point. To obtain the normal form, rescale the variables as

$$t_2\mapsto\frac{2}{\mu}\tau,\qquad \xi\mapsto\left(\frac{d_2+d_1}{\mu}\right)^{1/\gamma}x,$$
$$A\mapsto\sqrt{\frac{\mu}{2|\Re s|}}A,\qquad \alpha=\frac{d_2-d_1}{d_2+d_1}\frac{\nabla f_{11}}{\omega},\qquad \beta=\frac{\Im s}{\Re s},$$

$$\frac{\partial A}{\partial\tau}=A+(1+\alpha\mathrm{i})\mathfrak{D}^{\gamma}_{|x|}A-\operatorname{sign}(\Re s)(1+\beta\mathrm{i})A|A|^2. \tag{10}$$

If $\gamma_1<\gamma_2$, the expansion scales δ_j remain the same up to third order, where the anomalous term first appears: $\delta_j=\epsilon^j$, $j\in\{1,2,3\}$. The spatial scale choice is according to the activator exponent $\delta^{\gamma_1}=\epsilon^2$. Then $1+2\gamma_2/\gamma_1>3$ and the anomalous term of the inhibitor will be neglected at order δ_3. Actually, it will appear only at order $k+1$, where k is the greatest integer satisfying $1+2\gamma_2/\gamma_1\geqslant k$. The amplitude equation coincides with (10) upon setting $\gamma=\gamma_1$ and $d_2=0$. When $\gamma_1>\gamma_2$, the same holds with $\gamma=\gamma_2$ and $d_1=0$. Thus the fractional analogue of the complex Ginzburg–Landau equation is

$$\frac{\partial A}{\partial\tau}=A+(1+\alpha\mathrm{i})\mathfrak{D}^{\gamma}_{|x|}A-\operatorname{sign}(\Re s)(1+\beta\mathrm{i})A|A|^2,$$
$$\gamma=\min\{\gamma_1,\gamma_2\}. \tag{11}$$

This equation was formerly derived in [15] in the problem of nonlinear oscillators' dynamics with long range interactions. In the present paper, the properties of the super-critical version ($\Re s>0$) are studied.

3. Symmetry properties

The normal complex Ginzburg–Landau equation possesses translational space $x\to x+c_x$, time $\tau\to\tau+c_t$ and phase $A\to A\exp(\mathrm{i}c)$ symmetry. Moreover, solutions within the class of modulated waves

$$A(x,\tau)=B(r)\,\mathrm{e}^{\mathrm{i}(q\,x-\varpi\tau)},\qquad r=x-v\tau,\qquad q,\varpi,v\in\mathbb{R} \tag{12}$$

are connected by a similarity transformation within the family $(\alpha-\beta)/(1+\alpha\beta)=\text{const}$ [11]. With the introduction of anomaly, equation (11) loses the Galilean invariance, and the symmetry within the class of modulated waves is preserved only for $v=0$. A solution with given (α,β) connects to a solution with (α',β') by $B=aB'$, $r=br'$. Then $q'=b\,q$,

$$a^2b^\gamma=\frac{1+\alpha'\beta'}{1+\alpha\beta}\frac{1+\alpha^2}{1+\alpha'^2},$$
$$\varpi'=\alpha'-b^\gamma\frac{1+\alpha'^2}{1+\alpha^2}(\alpha-\varpi),$$
$$b^{-\gamma}=\frac{1+\alpha\alpha'+(\alpha-\alpha')\varpi}{1+\alpha^2}. \tag{13}$$

4. Variational formulation

For the special case $\alpha=\beta$ the normal complex Ginzburg–Landau equation can be obtained by variation of a functional [11]. Rotation $A\mapsto A\exp(-\mathrm{i}\beta\tau)$ gives

$$\frac{\partial A}{\partial\tau}=(1+\mathrm{i}\beta)(A+\mathfrak{D}^{\gamma}_{|x|}A-|A|^2A). \tag{14}$$

Then the functional must be adjusted to yield the fractional Laplacian operator:

$$\Upsilon=\int_{-\infty}^{\infty}U(x,\tau)\,\mathrm{d}x,$$
$$U=-|A|^2+\frac{1}{2}|A|^4-\frac{\sec(\pi\gamma/2)}{2\Gamma(2-\gamma)}\left\{\frac{\partial A^*}{\partial x}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{A(\zeta)\,\mathrm{d}\zeta}{|x-\zeta|^{\gamma-1}}\right.$$
$$\left.+\frac{1}{2}(1-\gamma)A\int_{-\infty}^{\infty}\frac{\partial A^*}{\partial\zeta}\frac{\operatorname{sign}(x-\zeta)}{|x-\zeta|^{\gamma}}\,\mathrm{d}\zeta+\text{c.c.}\right\}+c. \tag{15}$$

It is possible to choose the constant c so that Υ converges, as the function A is γ-fold differentiable. The first variation is

$$\delta\Upsilon=-\int_{-\infty}^{\infty}(A-A^2A^*+\mathfrak{D}^{\gamma}_{|x|}A)\delta A^*\,\mathrm{d}x+\text{c.c.} \tag{16}$$

Since

$$\frac{\partial A}{\partial \tau} = -(1+\mathrm{i}\beta)\frac{\delta U}{\delta A^*}, \tag{17}$$

all solutions of (14) decay asymptotically in time:

$$\frac{\partial \Upsilon}{\partial \tau} = \int_{-\infty}^{\infty} \frac{\partial U}{\partial \tau}\,\mathrm{d}x = -\frac{2}{1+\beta^2}\int_{-\infty}^{\infty}\left|\frac{\partial A}{\partial t}\right|^2 \mathrm{d}x < 0. \tag{18}$$

5. Stability of traveling waves

Traveling waves $A_q(x,\tau) = \rho\exp(\mathrm{i}(qx - \varpi\tau))$ are a sub-class of (12) and comprise an important solution family of (11). By

$$\mathfrak{D}^{\gamma}_{|x|}\mathrm{e}^{\mathrm{i}qx} = -|q|^{\gamma}\mathrm{e}^{\mathrm{i}qx}, \tag{19}$$

it is straightforward to show that

$$\rho^2 = 1 - |q|^{\gamma}, \quad \varpi = \beta - (\beta-\alpha)|q|^{\gamma}. \tag{20}$$

5.1. Spatially homogeneous oscillations

Linearizing equation (11) about A_0, i.e.

$$A = \mathrm{e}^{-\mathrm{i}\beta\tau}(1+u+\mathrm{i}v), \quad u, v \in \mathbb{R}, \quad |u|, |v| \ll 1, \tag{21}$$

splitting into a real system and neglecting nonlinear terms of u and v,

$$\frac{\partial}{\partial \tau}\begin{pmatrix} u \\ v \end{pmatrix} = -2\begin{pmatrix} 1 & 0 \\ \beta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}\mathfrak{D}^{\gamma}_{|x|}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{22}$$

The eigenvalues λ of a normal disturbance

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}\mathrm{e}^{\lambda\tau + \mathrm{i}kx} \tag{23}$$

satisfy

$$\lambda^2 + 2\lambda(1+|k|^{\gamma}) + |k|^{\gamma}\left((1+\alpha^2)|k|^{\gamma} + 2(1+\alpha\beta)\right) = 0. \tag{24}$$

Since $\lambda_1 + \lambda_2 = -2(1+|k|^{\gamma}) < 0$, the disturbance is unstable if

$$\lambda_1\lambda_2 = |k|^{\gamma}\left((1+\alpha^2)|k|^{\gamma} + 2(1+\alpha\beta)\right) < 0, \tag{25}$$

which gives a set of unstable wavenumbers when $1+\alpha\beta < 0$:

$$0 < |k| < k_m, \quad k_m^{\gamma} = -2\frac{1+\alpha\beta}{1+\alpha^2}. \tag{26}$$

Thus the instability domain in the (α, β) plane coincides with the normal Benjamin–Feir domain.

5.2. Arbitrary wave

The evolution of a small perturbation $a(x,\tau)$ about an arbitrary wave solution A_q is governed by

$$\frac{\partial a}{\partial \tau} = a + (1+\mathrm{i}\alpha)\mathfrak{D}^{\gamma}_{|x|}a - (1+\mathrm{i}\beta)(2a|A_q|^2 + a^*A_q^2). \tag{27}$$

Without loss of generality the base wave may be taken one-dimensional (1D). However, the disturbance should combine a longitudinal and transverse waves:

$$a = A_{q+k}(\tau)\mathrm{e}^{\mathrm{i}(q+k_x)x + \mathrm{i}k_y y} + A_{q-k}(\tau)\mathrm{e}^{\mathrm{i}(q-k_x)x - \mathrm{i}k_y y}. \tag{28}$$

The resulting system of equations is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} A_{q+k} \\ A^*_{q-k} \end{pmatrix} = \mathfrak{A}_q\begin{pmatrix} A_{q+k} \\ A^*_{q-k} \end{pmatrix}, \tag{29}$$

wherein $\mathfrak{A}_q$ is a 2×2 matrix whose entries depend on k_x and k_y. With $A_{q\pm k} = A_{\pm}\exp((\lambda \mp \mathrm{i}\varpi)\tau)$ the eigenvalues must satisfy a quadratic equation

$$\det\begin{pmatrix} \lambda - \mathrm{i}\varpi - \mathfrak{A}_{11} & -\mathfrak{A}_{12} \\ -\mathfrak{A}^*_{12} & \lambda + \mathrm{i}\varpi - \mathfrak{A}_{22} \end{pmatrix} = 0. \tag{30}$$

Below, some particular relations between the longitudinal and transverse disturbance wavenumbers are considered, and equation (30) is expanded appropriately at the corresponding limits.

Note that the underlying wave A_q is neutrally stable: for $k_x = k_y = 0$ the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -2(1-|q|^{\gamma}) < 0$, thus rendering the long wave disturbances of special interest. For small ratios k_x/q, k_y/q, one can distinguish between two qualitatively distinct cases:

$$\begin{array}{ll} \text{(i)} & O(k_x/q) \sim O(k_y/q) \sim O(\nu), \\ \text{(ii)} & O(k_x/q) \sim O(k_y^2/q^2) \sim O(\nu), \end{array} \quad \nu \ll 1. \tag{31}$$

Firstly, suppose that (31(i)) holds. Then the expansion is taken up to order $O(\nu^2)$, because at $O(\nu)$ the real part of λ_1 vanishes:

$$\Re\lambda_1 \sim \frac{\gamma}{2}|q|^{\gamma}\left(-(1+\alpha\beta)\left((\gamma-1)\frac{k_x^2}{q^2} + \frac{k_y^2}{q^2}\right) + (1+\beta^2)\frac{\gamma|q|^{\gamma}}{1-|q|^{\gamma}}\frac{k_x^2}{q^2}\right) + O(\nu^3). \tag{32}$$

Thus, if $1+\alpha\beta < 0$, instability is immediate for any wave (20). For $1+\alpha\beta > 0$, solving to leading order the inequality $\Re\lambda_1 > 0$ yields

$$\left(\frac{k_y}{k_x}\right)^2 < \frac{1+\beta^2}{1+\alpha\beta}\frac{\gamma|q|^{\gamma}}{1-|q|^{\gamma}} + 1 - \gamma, \tag{33}$$

wherein coercing positiveness of the right-hand side gives a subset of unstable wavenumbers

$$q_m < |q| < 1, \quad q_m^{-\gamma} = 1 + \frac{\gamma}{\gamma-1}\frac{1+\beta^2}{1+\alpha\beta}. \tag{34}$$

Then, in the (k_x, k_y) plane, the instability region is bounded by two intersecting straight lines. For traveling waves within this subset, pure longitudinal disturbances $(k_y = 0)$ have the highest growth rate, recovering the Eckhaus instability criterion for the normal Ginzburg–Landau equation.

Now, suppose (31(ii)) holds. In this case, an expansion to order $O(\nu)$ suffices:

$$\Re\lambda_1 \sim -\frac{\gamma}{2}|q|^{\gamma}(1+\alpha\beta)\frac{k_y^2}{q^2} + O(\nu^2). \tag{35}$$

Here the instability ensues only for $1+\alpha\beta < 0$.

6. Phase diffusion equation

At the opposite limit of small ratios $q/k_x, q/k_y$, no new instability criteria emerge. The spatially homogeneous oscillation A_0 is unstable within the same region $1+\alpha\beta<0$ with respect to disturbances (26). The evolution of perturbations near the domain boundary is described by a fractional nonlinear phase diffusion equation (the fractional analog the Kuramoto–Sivashinsky equation). Define $0<\epsilon\ll 1$ so that $1+\alpha\beta=-\epsilon$. By (26) the spatial coordinate scale is $\chi=\epsilon^{1/\gamma}x$. To find the appropriate temporal scale take $|k|^\gamma=K\epsilon$ and expand (24) in powers of ϵ. The resulting approximation is

$$\lambda_1 \sim \epsilon^2\left(K-\tfrac{1}{2}(1+\alpha^2)K^2\right)+O(\epsilon^3). \tag{36}$$

Hence, the temporal scale is $\tau_2=\epsilon^2\tau$. Using (6) and rewriting (11) with χ and τ_2,

$$\epsilon^2\frac{\partial A}{\partial \tau_2}=A+\epsilon\left(1-\frac{\mathrm{i}}{\beta}(1+\epsilon)\right)\mathfrak{D}^\gamma_{|\chi|}A-(1+\mathrm{i}\beta)|A|^2A. \tag{37}$$

Taking

$$A=\mathrm{e}^{-\mathrm{i}\beta\tau_2/\epsilon^2}r(\chi,\tau_2)\,\mathrm{e}^{\mathrm{i}\varphi(\chi,\tau_2)}, \tag{38}$$

where

$$r(\chi,\tau_2)=1+\sum_{j=1}^{\infty}\epsilon^j r_j(\chi,\tau_2), \qquad \varphi=\sum_{j=1}^{\infty}\epsilon^j\varphi_j(\chi,\tau_2), \tag{39}$$

substituting into (37), dividing by $\exp(\mathrm{i}\varphi)$ and using the expansions

$$\mathrm{e}^{\pm\mathrm{i}\varphi}=1\pm\mathrm{i}\sum_{j=1}^{\infty}\epsilon^j\varphi_j-\frac{1}{2}\left(\sum_{j=1}^{\infty}\epsilon^j\varphi_j\right)^2+\cdots, \tag{40}$$

it is possible to collect powers of ϵ. At order $O(\epsilon^3)$, the following equation is obtained:

$$\frac{\partial\varphi_1}{\partial\tau_2}=\frac{1}{2}\left(\beta+\frac{1}{\beta}\right)\left(-\frac{1}{\beta}\left(\mathfrak{D}^\gamma_{|\chi|}\right)^2\varphi_1+\mathfrak{D}^\gamma_{|\chi|}\varphi_1^2-2\varphi_1\mathfrak{D}^\gamma_{|\chi|}\varphi_1\right)-\mathfrak{D}^\gamma_{|\chi|}\varphi_1. \tag{41}$$

The operator $(\mathfrak{D}_{|\chi|})^2$ is defined in Fourier space by

$$\left(\mathfrak{D}^\gamma_{|\chi|}\right)^2\mathrm{e}^{\mathrm{i}q\chi}=|q|^{2\gamma}\,\mathrm{e}^{\mathrm{i}q\chi} \tag{42}$$

and cannot be related in a simple way to the operator $\mathfrak{D}^{2\gamma}_{|\chi|}$, because the order 2γ exceeds the range of definition of $\mathfrak{D}^\gamma_{|\chi|}$. Note that the coefficients of the linear terms $\mathfrak{D}^\gamma_{|\chi|}\varphi_1$ and $(\mathfrak{D}_{|\chi|})^2\varphi_1$ are consistent with (36). To bring (41) into parameter independent form, define

$$\tau=\tau_2 T,\qquad x=b\,\chi,\qquad \varphi=a\,\varphi_1,$$
$$b^\gamma=T=\frac{2\beta^2}{1+\beta^2},\qquad a=\beta+\frac{1}{\beta}. \tag{43}$$

Then

$$\frac{\partial\varphi}{\partial\tau}=-\mathfrak{D}^\gamma_{|x|}\varphi-(\mathfrak{D}^\gamma_{|x|})^2\varphi+\frac{1}{2}\mathfrak{D}^\gamma_{|x|}\varphi^2-\varphi\mathfrak{D}^\gamma_{|x|}\varphi. \tag{44}$$

Except for replacing the Laplacian by fractional operators, the resemblance to the Kuramoto–Sivashinsky nonlinear phase diffusion equation is obvious.

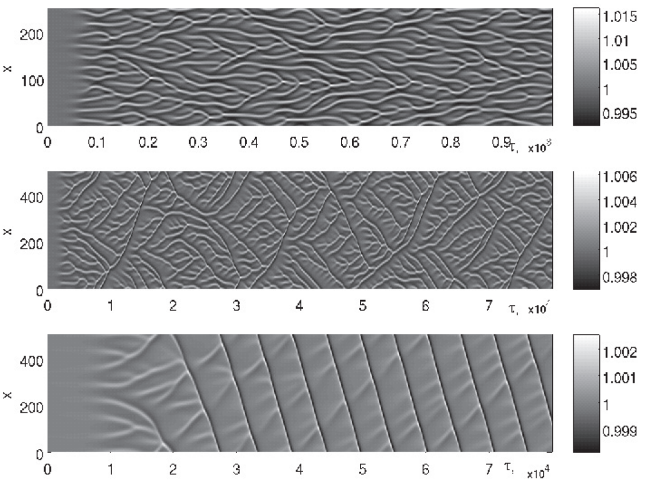

Figure 1. Amplitude $|A|$ in a phase turbulence regime as described by FCGL: spatio-temporal diagrams of numerical solutions of equation (11) for $\gamma=2.0$, $\beta=1.2$ (top), $\gamma=1.7$, $\beta=1.1$ (middle) and $\gamma=1.5$, $\beta=1.05$ (bottom); $\alpha=-1.0$.

7. Numerical simulations

The fractional complex Ginzburg–Landau equation (11) and fractional Kuramoto–Sivashinsky equation (44) have been solved numerically by a pseudo-spectral method with time integration in Fourier space, Crank–Nicolson scheme for linear operators and Adams–Bashforth scheme for nonlinear ones. Periodic boundary conditions, and if not specified otherwise, small amplitude random data as an initial condition have been used. Due to the extremely diverse dynamics of both CGL and KS equations [11, 16], this section is limited to the most interesting regimes that show the difference between the normal and fractional equations.

In one spatial dimension phase and amplitude turbulence, two regimes of complex behavior known for the normal CGLE [11] are comparable with the analogous dynamics of (11). Figure 1 shows phase turbulence regimes for different values of γ with α and β close to the Benjamin–Feir instability threshold. As the initial condition, a spatially homogeneous state $\mathrm{Re}A=\mathrm{const}$, $\mathrm{Im}A=\mathrm{const}$, $|A|=1$ was used with small amplitude random noise added. The upper subfigure corresponds to $\gamma=2.0$, i.e. normal CGL equation with its typical picture of phase turbulence and small amplitude modulations near $|A|=1$, exhibiting spatio-temporal chaos in the form of splitting and merging 'cells' [11]. This type of dynamics is described by the Kuramoto–Sivashinsky equation (see below). The middle subfigure shows a phase turbulence regime for $\gamma=1.7$. In this case, together with merging and splitting cells, long-living traveling shocks form, propagating in different directions. These shocks 'absorb' the adjacent 'cells'. The lower subfigure corresponds to $\gamma=1.5$. Here, a single shock forms in the whole domain and travels with a constant speed. The shock tails exhibit weak chaotic modulations. As shown below, all the above regimes can be captured by a fractional KS equation, describing phase dynamics near the Benjamin–Feir instability threshold. With increasing distance from the threshold, the shocks self-accelerate and trigger a transition to an amplitude turbulence regime, as shown in figure 2.

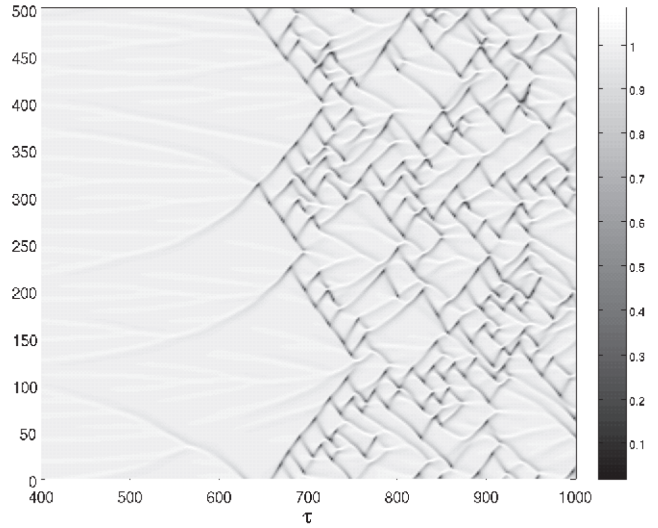

Figure 2. Amplitude $|A|$ during transition from phase turbulence to amplitude turbulence triggered by self-accelerating shocks: spatio-temporal diagrams of numerical solutions of equation (11) for $\gamma = 1.6$, $\beta = 1.3$, $\alpha = -1$.

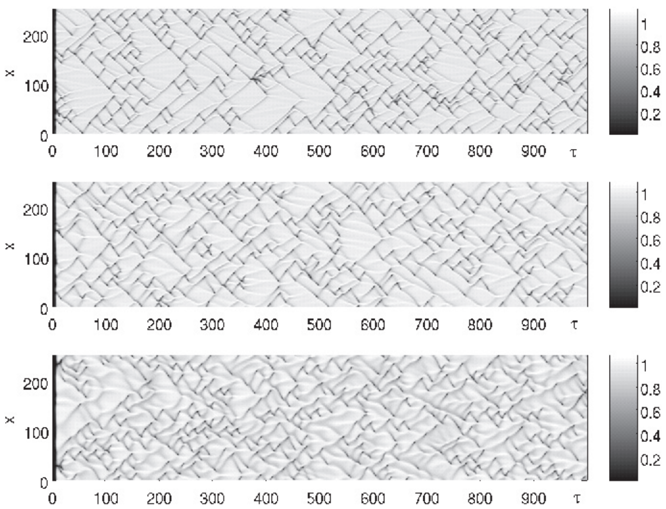

Figure 3. Amplitude A in amplitude turbulence regime as described by FCGLE: spatio-temporal diagrams of numerical solutions of equation (11) for $\gamma = 2.0$ (top), $\gamma = 1.5$ (middle), and $\gamma = 1.0$ (bottom); $\alpha = 1.0$, $\beta = -1.3$.

Figure 3 shows amplitude turbulence regimes for different values of γ. The usual amplitude turbulence dynamics comprising traveling hole solutions with a weak component of the phase turbulence is observed for $\gamma = 2$ (upper subfigure; see also [11]). As γ decreases, the phase turbulence component grows stronger, and for $\gamma = 1.0$ (lower subfigure) a combined phase-amplitude turbulent regime is formed with no traveling hole solutions.

Experimental observation of phase and amplitude turbulence regimes described by a 1D FCGL equation requires special arrangements in order to make the experimental system effectively 1D. More generic and easier to study in experiments are 2D systems, described by a 2D FCGL equation. Probably the most remarkable solution of a normal CGL equation in 2D is a spiral wave [11]. Therefore the effect of anomalous diffusion on spiral wave dynamics is of prime interest.

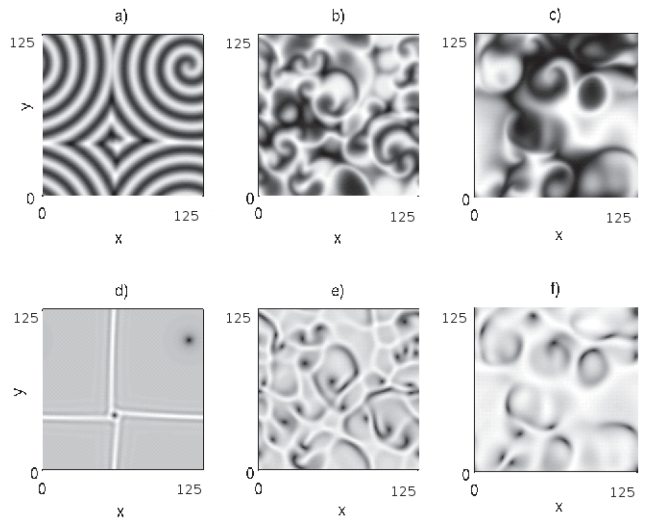

Figure 4. Solution snapshots of 2D FCGLE: $\Re A$ (upper) and $|A|$ (lower) for $\alpha = 1.5$, $\beta = -0.6$ and $\gamma = 1.9$ (a), (d); $\gamma = 1.8$ (b), (e); $\gamma = 1.05$ (c), (f).

Numerical simulations of the 2D FCGL equation have been performed for parameter values, corresponding to the formation of spiral waves in normal CGLE, by means of a similar 2D pseudo-spectral code. Figure 4 shows snapshots of the solutions for different values of γ. Figures 4(a) and (d) correspond to $\gamma = 1.9$. A single spiral wave is formed in the whole domain, akin to the normal case. In figures 4(b) and (e) ($\gamma = 1.8$) spiral-like defects form, where the core of each defect occupies a certain domain with domain walls between them. However, the domain walls in this case are partially dissolved and the defects (spiral cores) move in a chaotic manner. With further decrease of γ (figures 4(c) and (f); $\gamma = 1.05$), the number of defects decreases and the domain walls between them are almost completely dissolved. The system exhibits several domains with almost spatially homogeneous oscillations with different phases.

Finally, the fractional Kuramoto–Sivashinsky equation (44), describing phase dynamics near the Benjamin–Feir instability threshold, has been solved by a pseudo-spectral code with periodic boundary conditions and small amplitude random data as an initial condition. Figure 5 shows spatiotemporal solution diagrams for different values of γ. Figure 5(a), corresponding to normal diffusion ($\gamma = 2.0$), shows chaotic spatio-temporal dynamics of merging and splitting 'cells', as in phase turbulence of the normal CGLE shown in figure 1 (top). Figures 5(b) and (c) conform to $\gamma = 1.7$ and 1.6, respectively. Along with merging and splitting cells, traveling shocks appear, absorbing and emitting cells. As γ decreases, the shocks become more frequent and pronounced, and propagate faster, similarly to figure 1 (middle). When γ decreases below a certain, domain length dependent threshold, a single traveling shock is formed in the whole domain, as exemplified in figure 6: a spatio-temporal diagram (a) and a few solutions in successive moments of time (b). The shock travels with a constant speed, while its tails exhibit weak, chaotic spatio-temporal modulations. A similar shock formation is seen in figure 1 (bottom).

Note that the shock amplitude grows both with the decrease of γ and increase of the computational domain.

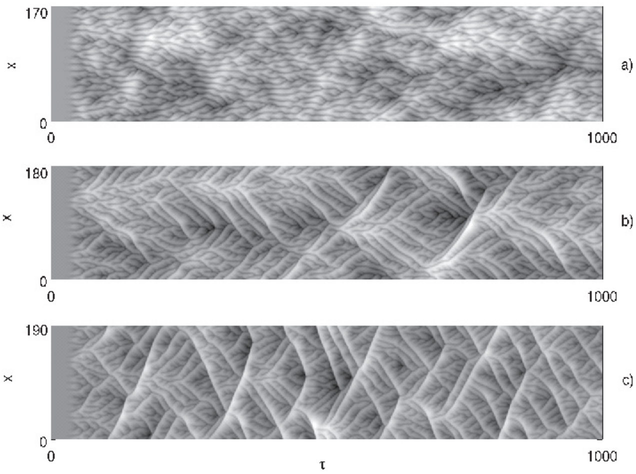

Figure 5. Spatio-temporal dynamics of FKS equation (44) for $\gamma = 2.0$ (a), $\gamma = 1.7$ (b) and $\gamma = 1.6$ (c).

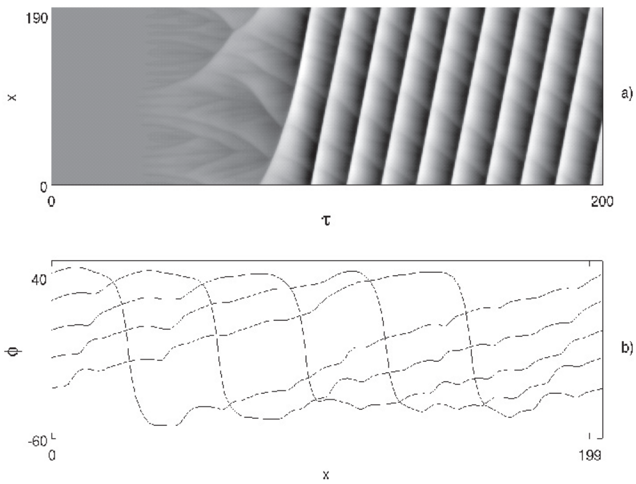

Figure 6. Solutions of FKS equation (44) for $\gamma = 1.5$: spatio-temporal diagram (a) and solutions at successive moments of time (b).

Below some threshold of γ, the shock accelerates with its amplitude growing exponentially. An asymptotic analysis revealed large amplitude asymptotic solutions of (44) in the form

$$\varphi = a(\tau) f(x - \zeta(\tau)), \qquad (45)$$

where f is an odd periodic function, $a(\tau)$ grows exponentially, and the instant speed $\mathrm{d}\zeta/\mathrm{d}\tau$ is proportional to $a(\tau)$, as confirmed by numerical simulations.

8. Discussion and conclusions

A reaction–diffusion system governed by Lévy flights has been reduced near a long wave (Hopf) bifurcation point. The reduced system is described by a complex Ginzburg–Landau (FCGL) equation with a fractional order Laplacian.

The fractional analogue does not inherit the Galilean invariance, a property well known for a normal Ginzburg–Landau equation, yet the similarity of modulated wave solutions along the family of curves $(\alpha - \beta)/(1 + \alpha\beta) = \text{constant}$ is retained. Another property common to normal and anomalous diffusion is the variational formulation in the special case $\alpha = \beta$.

Similarly to a normal complex Ginzburg–Landau equation, the fractional analogue possesses a family of solutions in the form of plane traveling waves. For a uniform oscillation the instability domain in α–β space coincides with the normal Benjamin–Feir domain. The instability with respect to 2D disturbances ensues for the whole unit circle within this domain and for a γ-dependent subset otherwise.

Near the Benjamin–Feir domain boundary the system dynamics is described by a fractional analogue of the Kuramoto–Sivashinsky equation (44) for the phase evolution.

Numerical solution of the 1D FCGL equation for decreasing values of the Lévy exponent γ reveals the appearance of traveling shock waves in the regime of phase turbulence, whereas amplitude turbulence exhibits a stronger phase turbulence component. The decrease of γ in 2D solutions leads to destruction of spiral waves and formation of defect chaos.

Solutions of the FKS equation for decreasing γ exhibit a transformation of chaotic dynamics of merging and splitting cells, typical of systems with normal diffusion, to traveling shocks. The same transition is observed for the FCGL equation. When γ decreases below a certain threshold, the shocks self-accelerate, their amplitude grows exponentially and the solution blows up. This phenomenon corresponds to the transition to amplitude turbulence and has been confirmed by numerical simulations.

Some remarks on the possible ways to control the anomalous diffusion effects in experiments are in order. For a reaction–diffusion system set up in a liquid layer with turbulent mixing, the means to control the Lévy exponent, characterizing turbulent diffusion, would be to control the mixing intensity and possibly the type of turbulent flow. For a reaction–diffusion system on a catalytic surface, in which the surface super-diffusion would be a consequence of the possibility for molecules to make long jumps over the surface through the gas phase, the Lévy exponent might be controlled by temperature change or control of the gas flow near the surface through affecting the adsorption bonds, say, by light irradiation.

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