Fractional amplitude and phase dynamics in super-diffusive reaction–diffusion systems

Y. Nec * [1], **A.A. Nepomnyashchy** [1], and **A.A. Golovin** [2]

[1] Department of Mathematics, Technion - Israel Institute of Technology, Haifa, Israel.
[2] Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, Illinois, USA.

Study of weakly non-linear dynamics of a reaction–super-diffusion system near a Hopf bifurcation by means of fractional analogues of complex Ginzburg-Landau and Kuramoto-Sivashinsky equations is presented.

Many physical processes of diffusive nature are not amenable to modelling by Fickian diffusion. A widely used description of super-diffusive transport relies on the continuous time random walk model [1] with a power law asymptotics of the particle jump length distribution, leading in the macroscopic limit to a diffusion equation with the Laplacian replaced by its *fractional power* [2]. Interplay of diffusion with non-linear reaction kinetics results in a quite involved phenomenon of pattern formation and spatio-temporal chaos. The purpose of the current study was to extend the generic equations describing such a system near a bifurcation point for the case of anomalous diffusion.

Consider a two-component reaction–diffusion system in the general case of distinct anomaly exponents:

$$\frac{\partial n_j}{\partial t} = d_j \mathfrak{D}^{\gamma_j}_{|x|} n_j + f_j(n_1, n_2), \quad j = \{1, 2\}, \tag{1}$$

where n_j, d_j and f_j are the species concentrations, diffusion coefficients and general kinetic functions, correspondingly. The fractional operator of order $1 < \gamma < 2$ is defined in physical and Fourier space as [3]

$$\mathfrak{D}^{\gamma}_{|x|} n(x) = -\frac{\sec(\pi\gamma/2)}{2\Gamma(2-\gamma)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \frac{n(\zeta)}{|x-\zeta|^{\gamma-1}} d\zeta, \qquad \mathfrak{D}^{\gamma}_{|\mathbf{x}|} e^{i\mathbf{q}\cdot\mathbf{x}} = -|\mathbf{q}|^{\gamma} e^{i\mathbf{q}\cdot\mathbf{x}}. \tag{2}$$

For a homogeneous steady state $\mathbf{n}_0$, i.e. $\mathbf{f}(\mathbf{n}_0) = \mathbf{0}$, Hopf bifurcation occurs at the long wave limit $q = 0$ when the trace of the sensitivity matrix $(\nabla \mathbf{f})_{jk} = \partial f_j / \partial n_k$ vanishes. At the threshold $\mathbf{n} = A(\xi, \tau)\mathbf{v} \exp(i\omega_0 t)$, where $\mathbf{v}, \omega_0$ and $A(\xi, \tau)$ are, correspondingly, an eigenvector, Hopf bifurcation frequency and complex amplitude depending on slow spatial and temporal scales ξ and τ. A multiple scales analysis yields the *fractional complex Ginzburg-Landau* (FCGL) equation:

$$\frac{\partial A}{\partial \tau} = A + (1 + \alpha i)\mathfrak{D}^{\gamma}_{|\xi|} A - (1 + \beta i) A|A|^2 \tag{3}$$

(in rescaled form). This equation was formerly derived in [4] in the problem of nonlinear oscillators' dynamics with long range interactions and is similar to a normal complex Ginzburg-Landau equation except that Laplacian is replaced by the fractional operator, whose order γ equals the smaller of the two species' exponents. The parameters α and β coincide with those of a normal reaction–diffusion system if $\gamma_1 = \gamma_2$, and are obtained by taking $d_2 = 0$ if $\gamma_1 < \gamma_2$ and $d_1 = 0$ if $\gamma_1 > \gamma_2$.

Equation (3) preserves the symmetries of a normal complex Ginzburg-Landau equation with respect to time and space translations and a phase change $A \mapsto A \exp(i\vartheta)$. Its solutions in the form $A(\xi, \tau) = B(\xi)\, e^{i(q\xi - \omega\tau)}$ have a symmetry akin to that in [5]. If a solution of this type is known for a pair (α, β), the solution for a new pair (α', β') located on one of the curves $(\alpha - \beta)/(1 + \alpha\beta) = \text{const}$ can be found by a similarity transformation $B = aB'$, $\xi = b\xi'$.

In the special case $\alpha = \beta$ after the phase shift $A \mapsto A \exp(-i\beta\tau)$, eq.(3), like a normal complex Ginzburg-Landau equation [6], can be written in a variational form,

$$\frac{\partial A}{\partial \tau} = -(1 + i\beta)\frac{\delta \Upsilon}{\delta A^*}, \quad \Upsilon = \int_{-\infty}^{\infty} U(\xi, \tau) d\xi, \tag{4}$$

$$U = -|A|^2 + \frac{|A|^4}{2} - \frac{\sec(\pi\gamma/2)}{\Gamma(2-\gamma)} \left\{ \frac{\partial A^*}{\partial \xi} \frac{\partial}{\partial \xi} \int_{-\infty}^{\infty} \frac{A(\zeta) d\zeta}{|\xi - \zeta|^{\gamma-1}} - \frac{1-\gamma}{2} A \int_{-\infty}^{\infty} \frac{\partial A^*}{\partial \zeta} \frac{\operatorname{sign}(\xi - \zeta)}{|\xi - \zeta|^{\gamma}} d\zeta + \text{c.c.} \right\} + c. \tag{5}$$

The constant c is chosen so that Υ converges. Then $\partial\Upsilon/\partial\tau = -2(1 + \beta^2)^{-1} \int_{-\infty}^{\infty} |\partial A/\partial t|^2 \, d\xi < 0$, and the system exhibits relaxational dynamics.

Now consider stability of the traveling wave solutions of (3),

$$A_q = \sqrt{1 - |q|^{\gamma}} e^{i(q\xi - \omega\tau)}, \ \omega = \beta - (\beta - \alpha)|q|^{\gamma}. \tag{6}$$

* Corresponding author: e-mail: flyby@techunix.technion.ac.il, Phone: +972 4 8294186

A small perturbation $a(\xi, \tau)$ about A_q comprises longitudinal and transverse waves of the form

$$a = A_{q+k}(\tau) e^{i(q+k_\xi)\xi + ik_\eta \eta} + A_{q-k}(\tau) e^{i(q-k_\xi)\xi - ik_\eta \eta}, \tag{7}$$

with k_ξ, k_η being the respective wave numbers. The solution (6) is neutrally stable with respect to disturbances $k_\xi = k_\eta = 0$. Further insight into long perturbations reveals that for $O(k_\xi/q) \sim O(k_\eta/q) \sim o(1)$ to leading order the growth rate of $A_{q\pm k} \sim \exp(\lambda\tau)$ satisfies

$$\Re\lambda \sim \frac{\gamma}{2}|q|^\gamma \left[-(1+\alpha\beta)\left((\gamma-1)\frac{k_\xi^2}{q^2} + \frac{k_\eta^2}{q^2} \right) + \gamma(1+\beta^2)\frac{|q|^\gamma}{1-|q|^\gamma}\frac{k_\xi^2}{q^2} \right]. \tag{8}$$

Therefore (6) is unstable if $1+\alpha\beta < 0$, i.e. Benjamin-Feir criterion for a normal CGLE is recovered. However, if $1+\alpha\beta > 0$, a γ-dependent set of unstable wave vectors exists, generalising Eckhaus instability criterion:

$$|q_m| < |q| < 1, \quad |q_m|^{-\gamma} = 1 + \frac{\gamma}{\gamma - 1}\frac{1+\beta^2}{1+\alpha\beta}. \tag{9}$$

No new instability criteria emerge in the opposite limit $q \ll k_\xi, k_\eta \ll 1$. In particular, the spatially-homogeneous oscillation $A_0 = \exp(-i\beta\tau)$ is unstable in the same region $1+\alpha\beta < 0$ with respect to disturbances whose wave numbers k satisfy

$$0 < |k|^\gamma < -\frac{2(1+\alpha\beta)}{(1+\alpha^2)}, \qquad 1+\alpha\beta < 0. \tag{10}$$

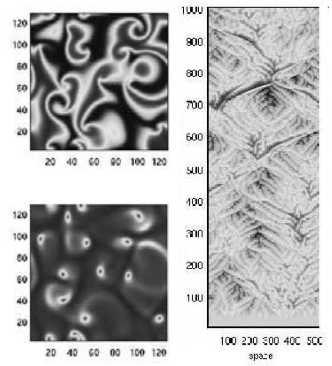
The evolution of perturbations near the curve $1+\alpha\beta = 0$ is expected to be described by an analogue of Kuramoto-Sivashinsky equation [7]. Reduction of (3) with a slowly evolving amplitude yields an equation for the phase ϕ:

$$\frac{\partial \phi}{\partial \tau} = -\mathfrak{D}^\gamma_{|\chi|}\phi - (\mathfrak{D}^\gamma_{|\chi|})^2\phi + \frac{1}{2}\mathfrak{D}^\gamma_{|\chi|}\phi^2 - \phi\mathfrak{D}^\gamma_{|\chi|}\phi. \tag{11}$$

The operator $(\mathfrak{D}^\gamma_{|\chi|})^2$ is defined in Fourier space by $(\mathfrak{D}^\gamma_{|\chi|})^2 e^{iq\chi} = |q|^{2\gamma} e^{iq\chi}$ and cannot be simply related to the operator $\mathfrak{D}^{2\gamma}_{|\chi|}$ as the order 2γ exceeds the definition range in (2). Eq.(11) is the *fractional Kuramoto-Sivashinsky equation* (FKS).

Numerical solutions of (3) and (11) revealed structures similar to those known for the normal counterparts, whose additional properties are to be studied in the future. Figure 1(a) depicts a defect chaos regime of a two dimensional variant of (3). Figure 1(b) depicts a spatio-temporal chaotic regime of (11) describing phase turbulence.

In conclusion, it has been shown that weakly non-linear dynamics of a super-diffusive reaction–diffusion system, characterised by Lévy flights, can be described by fractional analogues of complex Ginzburg-Landau and Kuramoto-Sivashinsky equations.



(a) $\Re A$ (upper) and $|A|$ (lower) (b) phase ϕ

Fig. 1 Chaotic regimes: (a) FCGL defect chaos, (b) FKS phase turbulence.

References

[1] E.W. Montroll and G.W. Weiss, J. Math. Phys. **6**, 167 (1965).

[2] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000); G.M. Zaslavsky, *Hamiltonian chaos and fractional dynamics*, 2005, Oxford University Press.

[3] I.M. Sokolov, A.V. Chechkin and J. Klafter, Physica A: Stat. Theor. Phys. **336**, 245 (2004).

[4] V.E. Tarasov and G.M. Zaslavsky, Chaos **16**, 023110 (2006).

[5] P.S. Hagan, SIAM J. Appl. Math. **42**, 762 (1982).

[6] I.S. Aranson and L. Kramer, Rev. Mod. Phys. **74**, 99 (2002).

[7] T. Bohr, M.H. Jensen, G. Paladin and A. Vulpiani, *Dynamical systems approach to turbulence*, 1998, Cambridge Univeristy Press.