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# Turing instability in sub-diffusive reaction–diffusion systems

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**Abstract**

Linear stability characteristics of an anomalous sub-diffusive activator–inhibitor system are investigated through a mesoscopic model, where anomaly ensues by a memory integro-differential operator with an equal anomaly exponent for all species. It is shown that monotonic instability, known from normal diffusion, persists in the anomalous system, thereby allowing Turing pattern occurrence. Presence of anomaly stabilizes the system by diminution of the range of diffusion coefficients' ratio involving instability. In addition, the maximal growth rate is diminished, proving the existence of an absolutely stable anomalous system.

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## 1. Introduction

It has been realized recently that in many physical processes the conception of normal diffusion, with its inherent temporal scaling of the mean square displacement $\langle r^2(t)\rangle \sim t$, is unsuitable. A more general relation, $\langle r^2(t)\rangle \sim t^\gamma$, is characteristic for sub-diffusion $(0 < \gamma < 1)$ or super-diffusion $(\gamma > 1)$. A variety of physical phenomena exhibiting anomalous diffusion can be found in the review papers [1, 2].

In a number of systems anomalous diffusion is accompanied by chemical reactions. For instance, biological media such as lipid bilayers hinder the reaction rate due to their involved structure [3]. In whole living cells the presence of numerous organelles also obstructs normal diffusion [4]. At a fractal interface such as an electrode–electrolyte the reaction rate is reduced due to surface irregularity [5].

The mathematical modelling of the reaction–diffusion phenomena in systems with anomalous diffusion is far from complete. The simplest replacement of the Laplacian operator by a fractional derivative with the reaction term unchanged may be reasonable when the reactions are activation limited or mediated by additional, normally diffusing reagents.

  

Conversely, if the progress of a diffusion limited reaction is impeded by the same physical factors that hinder the diffusion, the reaction term has to be modified by the application of a fractional derivative [6]. Still, there are memory effects that cannot be described by means of fractional derivative operators. As has been recently shown in [7–9] for a uni-molecular decay reaction, a more involved integro-differential operator is then required.

One of the basic phenomena observed in normal reaction–diffusion processes is the appearance of spatial (Turing) patterns, when a homogeneous steady state solution is linearly stable in the absence of diffusion, yet unstable to small spatial perturbations in the presence of diffusion [10–12].

In the present paper, the model developed in [7, 8] is extended to the case of an arbitrary number of species and arbitrary linear kinetics. This extension allows us to formulate the conditions for the appearance of Turing instability.

## 2. Mathematical model

A system of $n$ reacting species is considered, characterized by the vector of concentrations $\mathbf{N}(\mathbf{r}, t) = \{N_i(\mathbf{r}, t)\}, i = 1, \ldots, n$. When the species' spatial distribution is homogeneous, the temporal evolution of $\mathbf{N}$ is governed by the system of kinetic equations

$$\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}t} = \mathbf{f}(\mathbf{N}). \tag{1}$$

The function $\mathbf{f}$ is generally nonlinear (see [10] for specific examples). As the current paper is devoted to the effects of memory (due to sub-diffusion) on Turing instability, $\mathbf{f}$ is kept in its generic form.

If (1) possesses an equilibrium state $\mathbf{N}_0$ so that $\mathbf{f}(\mathbf{N}_0) = \mathbf{0}$, a small density perturbation $\mathbf{n}$ around $\mathbf{N}_0$ evolves according to

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t} = \nabla\mathbf{f}\,\mathbf{n}, \qquad (\nabla\mathbf{f})_{ij} = \frac{\partial f_i}{\partial N_j}, \qquad i, j \in \{1, \ldots, n\}. \tag{2}$$

Then by (2)

$$\mathbf{n}(t) = \mathrm{e}^{\nabla\mathbf{f}(t-t')}\mathbf{n}(t'), \tag{3}$$

where $t'$ is a reference time.

Later on, it is assumed that the eigenvalues of the matrix $(\nabla\mathbf{f})_{ij}$ have negative real parts corresponding to the stability of $\mathbf{N}_0$. However, it is known that in the presence of normal diffusion instability can arise with respect to spatially inhomogeneous disturbances (Turing instability) [10].

The appearance of Turing instability in a system with anomalous diffusion is considered in the framework of the following model. The basic assumption is that a mutual conversion of different species (on the background of the equilibrium state) is governed by the linear law (3) independently of their position and the time of their arrival to a definite position. This evolution law can be achieved as the result of chemical reactions mediated by some abundant and easily diffusing catalysts (rather than through a direct interaction of species). Assume also that the particles of species $i$ are subject to a continuous time random walk governed by the probability distribution function $\psi_i(\mathbf{r}, t) = \mu_i(\mathbf{r})w(t), i = 1, \ldots, n$, with the jump length distribution $\mu_i(\mathbf{r})$ dependent on the kind of molecules that carry out a jump, and the waiting time distribution independent of the kind of molecules (another model is discussed in the appendix).

As emphasized in [8], one can distinguish between two cases: (i) the chemical reaction does not affect the waiting time distribution, i.e., changes in chemical composition do not entail a change in particles' 'age'; (ii) the newborn particles produced by a reaction are assigned a new waiting time, i.e., zero 'age'. The choice should be according to the particular problem and underlying microscopic phenomena that are not discussed here. Case (ii) has been studied in [13, 14]. There, even in a nearly equilibrium state the diffusion might be essentially reduced due to persisting mutual transformations of particles, preventing their 'aging'. In the present paper, case (i) is chosen following [7, 8]: the particle 'age' is not changed due to the reaction, but the probability to jump is determined by the actual kind of the molecule at the time of the jump.

Now consider a 'population' of particles that belong to species $i$ and arrive at the location $\mathbf{r}$ at the time instant $t > 0$. Let $n_i(\mathbf{r}, t)$ be their density. Each particle from this population has carried out its previous jump to a certain location $\mathbf{r}'$ at a certain time instant $t'$. In the absence of the chemical reactions, the density $n_i(\mathbf{r}, t)$ is directly collected from all the points $(\mathbf{r}', t')$, hence

$$n_i(\mathbf{r}, t) = \int_\Omega \int_0^t \mu_i(\mathbf{r} - \mathbf{r}')w(t - t')n_i(\mathbf{r}', t')\,\mathrm{d}t'\,\mathrm{d}\mathbf{r}', \qquad t > 0. \tag{4}$$

The domain $\Omega$ is the whole space or has a rectangular shape; the dimension of the space is arbitrary. In the presence of a permanent change of the chemical composition governed by (3), which does not depend on the position and age, one has to replace the chemical composition $n_i(\mathbf{r}', t')$ on the right-hand side of (4) by the actual composition of particles at the time of the jump $t$, i.e., by

$$\sum_{j=1}^{n}(\mathrm{e}^{\nabla\mathbf{f}(t-t')})_{ij}n_j(\mathbf{r}', t').$$

Hence one arrives at the equation

$$n_i(\mathbf{r}, t) = \int_\Omega \int_0^t \mu_i(\mathbf{r} - \mathbf{r}')w(t - t')\sum_{j=1}^{n}(\mathrm{e}^{\nabla\mathbf{f}(t-t')})_{ij}n_j(\mathbf{r}', t')\,\mathrm{d}t'\,\mathrm{d}\mathbf{r}'. \tag{5}$$

To incorporate the initial condition, all the particles are taken with the initial density $\mathbf{n}_0(\mathbf{r})$ as arriving to the point $\mathbf{r}$ at the time instant $t = 0$. Using the vector form of presentation,

$$\mathbf{n}(\mathbf{r}, t) = \mathbf{n}_0(\mathbf{r})\delta(t) + \int_\Omega \int_0^t \mu(\mathbf{r} - \mathbf{r}')w(t - t')\,\mathrm{e}^{\nabla\mathbf{f}(t-t')}\mathbf{n}(\mathbf{r}', t')\,\mathrm{d}t'\,\mathrm{d}\mathbf{r}', \tag{6}$$

where $\mathbf{n}_0(\mathbf{r})$ is the initial density perturbation and $\delta(t)$ denotes the Dirac delta function. Here $\mu$ is a diagonal matrix with $\mu_{ii} = \mu_i$, $i = \{1, \ldots, n\}$. Fourier transform of (6) yields

$$\hat{\mathbf{n}}(\mathbf{q}, t) = \hat{\mathbf{n}}_0(\mathbf{q})\delta(t) + \hat{\mu}(\mathbf{q})\int_0^t w(t - t')\,\mathrm{e}^{\nabla\mathbf{f}(t-t')}\hat{\mathbf{n}}(\mathbf{q}, t')\,\mathrm{d}t', \tag{7}$$

where $\mathbf{q}$ is the wave vector and the hat denotes transformed functions. Later on, a long-wave approximation

$$\hat{\mu}_i(q) \sim 1 - q^2\sigma_i^2 + o(q^2), \qquad q = |\mathbf{q}|, \tag{8}$$

will be used. Laplace transform of (7) leads to an algebraic system,

$$\tilde{\hat{\mathbf{n}}}(\mathbf{q}, s)(I - \hat{\mu}(q)\mathcal{L}[w(t)\,\mathrm{e}^{\nabla\mathbf{f}t}](s)) = \hat{\mathbf{n}}_0(\mathbf{q}), \tag{9}$$

where $s$ is the transform variable, the tilde denotes transformed functions and $I$ is the identity matrix of appropriate dimension. Thus the disturbance dispersion relation can be written as

$$\det(I - \hat{\mu}(q)\mathcal{L}[w(t)\,\mathrm{e}^{\nabla\mathbf{f}t}](s)) = 0. \tag{10}$$

## 3. Dispersion relation

### *3.1. General formulae*

An explicit derivation of the dispersion relation from expression (10) requires some preliminary definitions. A non-singular matrix $A_{n\times n}$ is diagonizable in the basis of its eigenvectors, so that

$$A = V\Lambda V^{-1}, \qquad V = [\mathbf{v}_1 \cdots \mathbf{v}_n], \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}. \tag{11}$$

Bold font is used throughout for vectors, $\lambda_j$ are the eigenvalues of $A$ corresponding to the eigenvectors $\mathbf{v}_j$, $j = \{1, \ldots, n\}$. The rows of the inverse matrix $V^{-1}$ are denoted $\mathbf{r}_j$. The matrices $(-A)$, $\exp(A)$ and $A^\alpha$, $\alpha \in \mathbb{R}$, possess an identical set of eigenvectors, and the corresponding eigenvalues are, respectively, negatives, exponents and $\alpha$-powers of the eigenvalues of $A$. The scalar Laplace transform argument shift formula

$$\mathcal{L}[\mathrm{e}^{at}y(t)](s) = \mathcal{L}[y(t)](s-a) \tag{12}$$

is generalized as follows:

$$\begin{aligned}\mathcal{L}[\mathrm{e}^{At}\mathbf{y}(t)](s) &= \int_0^\infty \mathrm{e}^{-st}\,\mathrm{e}^{At}\mathbf{y}(t)\,\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-st}V\,\mathrm{e}^{\Lambda t}V^{-1}\mathbf{y}(t)\,\mathrm{d}t = \sum_{j=1}^n \mathbf{v}_j\mathbf{r}_j\mathcal{L}[\mathbf{y}(t)](s-\lambda_j).\end{aligned} \tag{13}$$

Henceforth $A = \nabla\mathbf{f}$, and the matrices $\Lambda$,$V$ are the corresponding diagonal matrix of eigenvalues and basis of eigenvectors of dimension $n$.

### *3.2. Particular cases*

*3.2.1. Turing instability for normal diffusion and super-diffusion.* Taking a two-species system with a waiting distribution function of the form

$$w(t) = \tau^{-1}\,\mathrm{e}^{-t/\tau} \tag{14}$$

and sensitivity matrix $\nabla\mathbf{f}$, whose distinct eigenvalues $\lambda_1, \lambda_2$ possess a negative real part, (10) becomes

$$\det\left(I - \tau^{-1}\mu\int_0^\infty \mathrm{e}^{(\nabla\mathbf{f}-(s+1/\tau)I)t}\,\mathrm{d}t\right) = 0. \tag{15}$$

Performing the integration and substituting

$$\mu \sim 1 - q^2\Sigma + o(q^2), \qquad \Sigma = \begin{pmatrix}\sigma_1^2 & 0 \\ 0 & \sigma_2^2\end{pmatrix}, \tag{16}$$

the dispersion relation known for normal diffusion is recovered:

$$\det(sI - \nabla\mathbf{f} - q^2 D) = 0, \qquad D = \tau^{-1}\Sigma. \tag{17}$$

In the case of super-diffusion caused by the long tails of the step length distribution function (Lévy flights),

$$\mu \sim 1 - |q|^\gamma\Sigma + o(|q|^\gamma), \qquad 1 < \gamma < 2, \tag{18}$$

and, in a similar way, one obtains

$$\det(sI - \nabla\mathbf{f} - |q|^\gamma D) = 0. \tag{19}$$

*3.2.2. Anomalous diffusion with exponential decay.* As a second example, take a single species system ($n = 1$) where $\exp(\nabla \mathbf{f})t$ is replaced by $\exp(-kt)$, while the Laplace transformed waiting distribution function corresponds to an anomalous diffusion [7, 8]:

$$\tilde{w}(s) \simeq 1 - \Gamma(1-\gamma)\tau^{\gamma} s^{\gamma} + o(|s|^{\gamma}). \tag{20}$$

Then

$$\mathcal{L}[w(t)\,\mathrm{e}^{-kt}] = \tilde{w}(s+k) \simeq 1 - \Gamma(1-\gamma)\tau^{\gamma}(s+k)^{\gamma} + o(|s|^{\gamma}), \tag{21}$$

and (10) becomes (to leading order)

$$s + k = -\frac{q^2\sigma^2}{\Gamma(1-\gamma)\tau^{\gamma}}(s+k)^{1-\gamma}, \tag{22}$$

recovering the results obtained in [7, 8].

*3.2.3. Anomalous diffusion with linear kinetics of arbitrary dimension n.* As in the previous example, $\tilde{w}(s)$ is given by (20). By (13),

$$\begin{aligned}\mathcal{L}[w(t)\,\mathrm{e}^{\nabla \mathbf{f} t}] &= \sum_{j=1}^{n} \mathbf{v}_j \mathbf{r}_j (1 - \Gamma(1-\gamma)\tau^{\gamma}(s-\lambda_j)^{\gamma}) \\ &= I - \Gamma(1-\gamma)\tau^{\gamma}(sI - \nabla \mathbf{f})^{\gamma}.\end{aligned} \tag{23}$$

Substituting into (10) and using (16) gives in the limit of small $q$

$$\det(q^2 C + (Is - \nabla \mathbf{f})^{\gamma}) = 0, \qquad C = \frac{\Sigma}{\Gamma(1-\gamma)\tau^{\gamma}}. \tag{24}$$

Another model leading to (24) is presented in the appendix.

### 3.3. *Stability of an activator–inhibitor anomalous system* ($n = 2$)

For the purpose of roots comparison with normal diffusion, (24) is written in an equivalent, more convenient form via multiplication by $(Is - \nabla \mathbf{f})^{1-\gamma}$ from the right:

$$\det(Is - \nabla \mathbf{f} + q^2 C V (Is - \Lambda)^{1-\gamma} V^{-1}) = 0. \tag{25}$$

Later on, it is assumed that $\lambda_1$ and $\lambda_2$, the eigenvalues of $\nabla \mathbf{f}_{2\times 2}$, are complex conjugate,

$$\lambda_1 = \tfrac{1}{2}(\mathrm{tr}\,\nabla \mathbf{f} + \mathrm{i}\sqrt{4\det \nabla \mathbf{f} - \mathrm{tr}^2 \nabla \mathbf{f}}), \qquad \lambda_2 = \lambda_1^*, \tag{26}$$

and $\mathrm{Re}\,\lambda_j < 0$ (the system is stable in the absence of diffusion). The eigenvectors basis is taken as

$$V = \begin{pmatrix} \nabla f_{12} & \nabla f_{12} \\ -(\nabla f_{11} - \lambda_1) & -(\nabla f_{11} - \lambda_2) \end{pmatrix}. \tag{27}$$

After rearrangement (25) becomes

$$s^2 - \mathrm{tr}\,\nabla \mathbf{f}\, s + \det \nabla \mathbf{f} + d q^4 (s-\lambda_1)^{1-\gamma}(s-\lambda_2)^{1-\gamma} + \frac{q^2}{\det V}\nabla f_{12}(S(s,\lambda_1) - S(s,\lambda_2)) = 0 \tag{28a}$$

with

$$S(s,\lambda) = s\mathcal{G}(s,\lambda) + \mathcal{F}(s,\lambda), \tag{28b}$$

$$\mathcal{F}(s,\lambda) = [(d+1)\det \nabla \mathbf{f} - \lambda^* \mathrm{tr}_w \nabla \mathbf{f}](s-\lambda)^{1-\gamma}, \tag{28c}$$

$$\mathcal{G}(s,\lambda) = [(d-1)\nabla f_{11} + \lambda^* - d\lambda](s-\lambda)^{1-\gamma}, \tag{28d}$$

and the sensitivity matrix weighted trace being

$$\mathrm{tr}_w \nabla \mathbf{f} = d\nabla f_{11} + \nabla f_{22}. \tag{29}$$

For $\gamma = 1$ expression (28*a*) immediately reduces to the relation known for normal diffusion. Then with $s = 0$ the quartics

$$dq^4 - \mathrm{tr}_w \nabla \mathbf{f}\, q^2 + \det \nabla \mathbf{f} = 0 \tag{30}$$

gives the set of coefficients $d$ for instability and the corresponding range of unstable wave numbers. For $\gamma < 1$ it is replaced by

$$d(\det \nabla \mathbf{f})^{1-\gamma} q^4 - q^2 \frac{\mathrm{Im}\,\mathcal{F}(0,\lambda_1)}{\mathrm{Im}\,\lambda_1} + \det \nabla \mathbf{f} = 0. \tag{31}$$

As the system is stable in the absence of diffusion, $\det \nabla \mathbf{f} > 0$ [10]. Hence the roots' product is positive,

$$q_+^2 q_-^2 = \frac{1}{d}(\det \nabla \mathbf{f})^{\gamma} > 0, \tag{32}$$

and their real parts are of identical sign. To express $\mathrm{Im}\,\mathcal{F}(0,\lambda_1)$ the following notation is introduced:

$$\lambda_1 = \sqrt{\det \nabla \mathbf{f}}\, \mathrm{e}^{\mathrm{i}\theta}, \qquad \frac{\pi}{2} < \theta < \pi, \qquad \vartheta = \pi - \theta. \tag{33}$$

Then

$$\mathrm{Im}\,\mathcal{F}(0,\lambda_1) = (\det \nabla \mathbf{f})^{1-\gamma/2}(\mathrm{tr}_w \nabla \mathbf{f} \sin \gamma\vartheta - (d+1)\sqrt{\det \nabla \mathbf{f}} \sin(1-\gamma)\vartheta). \tag{34}$$

Since $\mathrm{Im}\,\lambda_1 > 0$,

$$\mathrm{sign}(q_+^2 + q_-^2) = \mathrm{sign}\,\mathrm{Im}\,\mathcal{F}(0,\lambda_1). \tag{35}$$

For the roots to have positive real parts

$$\mathrm{tr}_w \nabla \mathbf{f} > (d+1)\sqrt{\det \nabla \mathbf{f}}\, f(\gamma;\vartheta), \qquad f(\gamma;\vartheta) = \frac{\sin(1-\gamma)\vartheta}{\sin \gamma\vartheta}. \tag{36}$$

The function $f(\gamma;\vartheta)$ decreases monotonically over the range $0 \leqslant \gamma \leqslant 1$:

$$f(0;\vartheta) = \infty, \qquad f(1;\vartheta) = 0, \qquad \frac{\partial f}{\partial \gamma} = -\vartheta \frac{\sin \vartheta}{\sin^2 \gamma\vartheta} < 0. \tag{37}$$

Therefore inequality (36) poses an increasingly severe condition relatively to the normal $\mathrm{tr}_w \nabla \mathbf{f} > 0$, as the system becomes more anomalous. The emanating normal demand $\nabla f_{11} > 0$ is replaced by a consequently more severe requirement

$$\nabla f_{11} > \sqrt{\det \nabla \mathbf{f}}\, f(\gamma;\vartheta). \tag{38}$$

For the roots to be real the discriminant of (31) should be positive, giving

$$\mathrm{tr}_w^2 \nabla \mathbf{f}\, g(\gamma;\vartheta) - 4d \det \nabla \mathbf{f} > h(\gamma;\nabla \mathbf{f}, d), \tag{39a}$$

$$g(\gamma;\vartheta) = \frac{\sin^2 \gamma\vartheta}{\sin^2 \vartheta} \tag{39b}$$

$$h(\gamma;\nabla \mathbf{f}, d) = 2\mathrm{tr}_w \nabla \mathbf{f}(d+1)\sqrt{\det \nabla \mathbf{f}} \frac{\sin \gamma\vartheta}{\sin^2 \vartheta} \sin(1-\gamma)\vartheta - (d+1)^2 \det \nabla \mathbf{f} \frac{\sin^2(1-\gamma)\vartheta}{\sin^2 \vartheta}. \tag{39c}$$

The function $g(\gamma;\vartheta)$ grows monotonically over the range $0 \leqslant \gamma \leqslant 1$:

$$g(0;\vartheta) = 0, \qquad g(1;\vartheta) = 1, \qquad \frac{\partial g}{\partial \gamma} = \vartheta \frac{\sin 2\gamma\vartheta}{\sin^2\vartheta} > 0. \tag{40}$$

The function $h(\gamma; \nabla\mathbf{f}, d)$ is not monotonic. Seeking an extremum gives

$$\tan(1-2\gamma_{\mathrm{ext}})\vartheta = -\frac{(d+1)\sqrt{\det\nabla\mathbf{f}}\sin\vartheta}{2\,\mathrm{tr}_w\nabla\mathbf{f} + (d+1)\sqrt{\det\nabla\mathbf{f}}\cos\vartheta}. \tag{41}$$

The right-hand side is negative, thus $\gamma_{\mathrm{ext}} > 1/2$. Solving $h(\gamma;\nabla\mathbf{f}, d) = 0$ yields $\gamma = 1$ and another root $\gamma_*$ satisfying

$$\cot\gamma_*\vartheta = \cot\vartheta + \frac{2\,\mathrm{tr}_w\nabla\mathbf{f}}{(d+1)\sqrt{\det\nabla\mathbf{f}}\sin\vartheta}, \tag{42}$$

whence

$$\cot\gamma_*\vartheta > \cot\vartheta. \tag{43}$$

Since cotangent is a descending function, $\gamma_* < 1$. Therefore $h(\gamma;\nabla\mathbf{f}, d)$ ascends from some negative value at $\gamma = 0$, crosses the abscissa at $\gamma_*$, attains a maximum at $\gamma_{\mathrm{ext}}$ and descends again to zero at $\gamma = 1$. Inspection of (39*a*) leads to the following conclusion: as $\gamma$ becomes slightly smaller than unity, the right-hand side of (39*a*) turns positive through $h(\gamma;\nabla\mathbf{f}, d)$, whereas the left-hand side grows smaller via $g(\gamma;\vartheta)$. Therefore condition (39*a*) is more severe than the normal,

$$\mathrm{tr}_w^2\nabla\mathbf{f} - 4d\det\nabla\mathbf{f} > 0. \tag{44}$$

Moreover, at some range of $0 < \gamma < 1$, this condition will become unsatisfiable if the left-hand side turns negative whilst $h(\gamma;\nabla\mathbf{f}, d)$ still remains positive. As $\gamma$ is diminished further, it will be possible to satisfy (39*a*) again. To have an estimation of this range of $\gamma$, note that at $\gamma = \gamma_*$ condition (36) fails. To see this, rearrange inequality (36) into

$$\cot\gamma\vartheta < \cot\vartheta + \frac{\mathrm{tr}_w\nabla\mathbf{f}}{(d+1)\sqrt{\det\nabla\mathbf{f}}\sin\vartheta} \tag{45}$$

and compare to (42). From cotangent monotonic descent it follows that to satisfy (36) the value of $\gamma$ should exceed to some extent $\gamma_*$.

Similarly to the normal case, it is possible to find the critical ratio of diffusion coefficients by equating the discriminant of (31) to zero. The implicit relation is

$$d_M(\gamma) = \frac{1}{4\det\nabla\mathbf{f}}\frac{1}{\sin^2\vartheta}(\mathrm{tr}_w\nabla\mathbf{f}\sin\gamma\vartheta - (d_M+1)\sqrt{\det\nabla\mathbf{f}}\sin(1-\gamma)\vartheta)^2, \tag{46}$$

a quadratics for $d_M(\gamma)$. Differentiating with respect to $\gamma$ and rearranging yields

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}d_M(\gamma) = -\vartheta\frac{\mathrm{tr}_w\nabla\mathbf{f}\cos\gamma\vartheta + \sqrt{\det\nabla\mathbf{f}}(d_M+1)\cos(1-\gamma)\vartheta}{\nabla f_{11}\sin\gamma\vartheta - \sqrt{\det\nabla\mathbf{f}}(\sin(1-\gamma)\vartheta + \sin\vartheta/\sqrt{d_M})}. \tag{47}$$

By (46)

$$\begin{aligned} d_M(\nabla f_{11}\sin\gamma\vartheta - \sqrt{\det\nabla\mathbf{f}}\sin(1-\gamma)\vartheta) + \nabla f_{22}\sin\gamma\vartheta - \sqrt{\det\nabla\mathbf{f}\sin(1-\gamma)\vartheta} \\ = 2\sqrt{d_M}\sqrt{\det\nabla\mathbf{f}}\sin\vartheta, \end{aligned} \tag{48}$$

so that

$$d_M(\nabla f_{11}\sin\gamma\vartheta - \sqrt{\det\nabla\mathbf{f}}\sin(1-\gamma)\vartheta) > 2\sqrt{d_M}\sqrt{\det\nabla\mathbf{f}}\sin\vartheta \tag{49}$$

as $\nabla f_{22} < 0$ to keep $\lambda_{1,2}$ stable whilst $\nabla f_{11} > 0$, and in particular the denominator in (47) is positive by (38). Then, recalling the range of $\vartheta$,

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}d_M(\gamma) < 0 \qquad \text{at} \qquad \gamma_* < \gamma < 1, \tag{50}$$

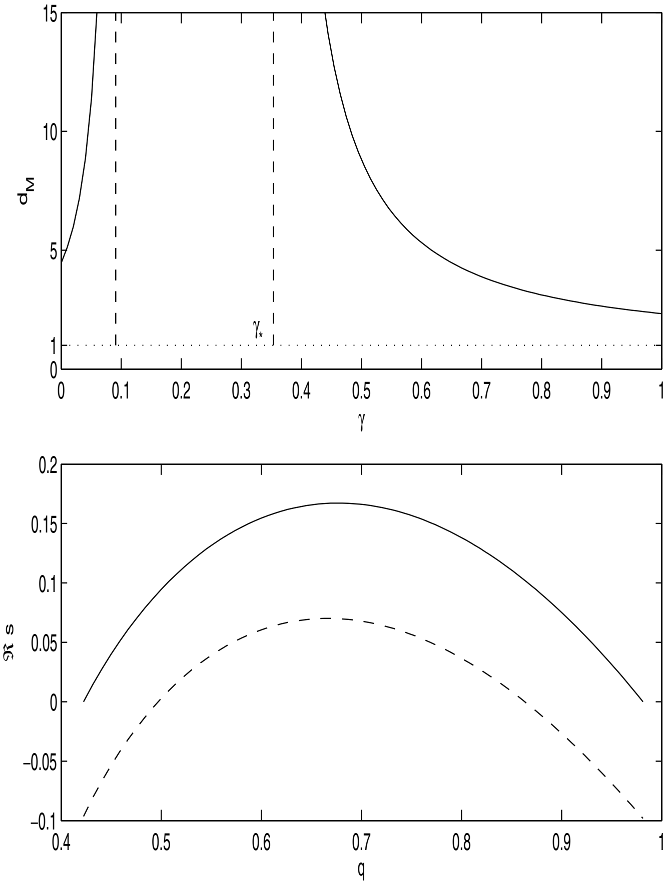


**Figure 1.** Upper: anomalous critical diffusion coefficients' ratio for $\nabla f_{11} = 2, \nabla f_{22} = -2.5$, $\det \nabla \mathbf{f} = 0.5$ (solid). Dashed lines mark the interval of stability (formally, $d_M < 0$). The dotted line marks the physical limit $d = 1$. Note that $d_M(\gamma) > d_M(1)$. Each point $(\gamma > \gamma_*, d_M)$ on the plot corresponds to a system with a negative growth rate curve, tangent to the real axis at a single point $q = q_- = q_+$. For the gradual descent of the curve below the real axis, see the lower picture. Lower: the growth rate versus the wave number for $\gamma = 1$ (solid), $\gamma = 0.9$ (dashed).

proving that the anomalous critical ratio is higher than the normal value $d_M(1)$, rendering the anomalous system always more stable than normal. Figure 1 depicts a typical example. Note that there exists an interval of $\gamma$ below the root $\gamma_*$, where the anomalous $d_M$ attains unphysical negative values, i.e., the system is stable for all $d$. This interval corresponds to the inability to satisfy (39*a*), as mentioned above. For values $\gamma$ close to zero the condition is satisfiable again, still keeping $d_M(\gamma) > d_M(1)$. However, this interval does not entail instability as the additional condition (36) fails for all values $0 < \gamma < \gamma_*$ and for some range above it.

From the standpoint of physics, the appearance of instability in the system above the threshold of $d_M(\gamma)$ will lead to a bifurcation and emergence of a different stable solution. Linear analysis predicts that threshold; however, all bifurcation properties (e.g. sub- or supercriticality) can be determined only via nonlinear analysis. However, a nonlinear generalization of the model is a difficult task, because there arises a question about the 'age' of all reaction products, a question that cannot be resolved in the framework of a general consideration. An example of a nonlinear model appears in [14], where the assumptions concerning the particles' 'age' differ essentially from the current work.

### 3.4. Eigenvalues near the ends of the instability interval

The set of unstable wave numbers ought to be determined and compared with the normal case. For this purpose the growth rate function behaviour in close vicinity of the roots $q = q_\pm$ is necessary. Expansion of (28*a*) about $q_\pm^2$ with $|s| \ll 1, \lambda_1, \lambda_2 \sim O(1)$ yields

$$Ps + Q \sim 0 + o(|s|), \tag{51a}$$

$$Q = d(\det \nabla \mathbf{f})^{1-\gamma} q^4 - q^2 \frac{\operatorname{Im} \mathcal{F}(0, \lambda_1)}{\operatorname{Im} \lambda_1} + \det \nabla \mathbf{f}, \tag{51b}$$

$$P = -\operatorname{tr} \nabla \mathbf{f}(1 + dq^4(1-\gamma)(\det \nabla \mathbf{f})^{-\gamma}) - q^2 \frac{\operatorname{Im} \mathcal{G}(0, \lambda_1) + (1-\gamma)\operatorname{Im} \mathcal{H}}{\operatorname{Im} \lambda_1}, \tag{51c}$$

$$\mathcal{H} = [(d+1) \det \nabla \mathbf{f} - \lambda_2 \operatorname{tr}_w \nabla \mathbf{f}](-\lambda_1)^{-\gamma}. \tag{51d}$$

Note that the quotient $\mathcal{H}$ equals $\mathcal{F}(0, \lambda_1)$ upon replacing $\gamma$ by $\gamma + 1$. As the sign of $Q$ depends on $d$, $Q > 0$ for all $q$ or for the set $\{q \,|\, q < q_- \text{ or } q > q_+\}$. To determine the sign of $P$, denote

$$\begin{aligned} y(\gamma; \nabla \mathbf{f}, d) = {} & (d-1)\nabla f_{11} \sin(1-\gamma)\vartheta + \sqrt{\det \nabla \mathbf{f}}(d \sin(2-\gamma)\vartheta + \sin \gamma\vartheta) \\ & - (1-\gamma)(\operatorname{tr}_w \nabla \mathbf{f} \sin(\gamma+1)\vartheta + (d+1)\sqrt{\det \nabla \mathbf{f}} \sin \gamma\vartheta). \end{aligned} \tag{52}$$

Then

$$P = \underbrace{-\operatorname{tr} \nabla \mathbf{f}(1 + dq^4(1-\gamma)(\det \nabla \mathbf{f})^{-\gamma})}_{>0} + q^2 y(\gamma; \nabla \mathbf{f}, d) \frac{(\det \nabla \mathbf{f})^{-\gamma/2}}{\sin \vartheta}. \tag{53}$$

Since at large $\gamma$ the function $y(\gamma; \nabla \mathbf{f}, d) > 0$, it may have two roots, both lying within or outside the set $\{\gamma \,|\, 0 < \gamma < 1\}$ or it may have no roots, as

$$\begin{aligned} y(0; \nabla \mathbf{f}, d) &= -\operatorname{tr} \nabla \mathbf{f} \sin \vartheta + d\sqrt{\det \nabla \mathbf{f}} \sin 2\vartheta > 0, \\ y(1; \nabla \mathbf{f}, d) &= (d+1)\sqrt{\det \nabla \mathbf{f}} > 0. \end{aligned} \tag{54}$$

Solving $y(\gamma^\dagger; \nabla \mathbf{f}, d) = 0$ yields

$$\cot \gamma^\dagger \vartheta = \cot \vartheta + c(\gamma^\dagger; \nabla \mathbf{f}, d), \tag{55}$$

where

$$c(\gamma^\dagger; \nabla \mathbf{f}, d) = \frac{2(1-\gamma^\dagger)\operatorname{tr}_w \nabla \mathbf{f} \cos \vartheta - \gamma(d+1)\sqrt{\det \nabla \mathbf{f}}}{\sin \vartheta [\gamma^\dagger \operatorname{tr}_w \nabla \mathbf{f} - \operatorname{tr} \nabla \mathbf{f} + 2d\sqrt{\det \nabla \mathbf{f}} \cos \vartheta]}. \tag{56}$$

By

$$c(0; \nabla \mathbf{f}, d) > 0, \qquad c(1; \nabla \mathbf{f}, d) < 0 \tag{57}$$

the function $c(\gamma^\dagger; \nabla \mathbf{f}, d)$ changes sign at

$$\gamma^\ddagger = \left(1 + \frac{(d+1)\sqrt{\det \nabla \mathbf{f}}}{2 \operatorname{tr}_w \nabla \mathbf{f} \cos \vartheta}\right)^{-1} < 1. \tag{58}$$

Hence for $\gamma^\ddagger < \gamma < 1$ the function $c(\gamma^\ddagger; \nabla \mathbf{f}, d) < 0$, and the root of $y(\gamma^\dagger; \nabla \mathbf{f}, d)$ satisfies $\gamma^\dagger > 1$. Then $y(\gamma; \nabla \mathbf{f}, d) > 0$ at $0 < \gamma < 1$, proving $P > 0$. Since $Q < 0 \,\forall q_- < q < q_+$, the growth rate $s \sim -Q/P$ is positive for the wave numbers within this range close to $q_\pm$ (wherever $|s| \ll 1$ holds). Conversely, for $0 < \gamma < \gamma^\ddagger$ the function $c(\gamma^\ddagger; \nabla \mathbf{f}, d) > 0$, and

$y(\gamma; \nabla\mathbf{f}, d) < 0$ for some range within $0 < \gamma < 1$. Then $P$ may or may not become negative due to its additional positive term. If $P$ remains positive, the situation is as above. If $P$ turns negative, then for the corresponding range of $\gamma$ $s > 0$ for wave numbers outside $(q_-, q_+)$, close to $q_\pm$. In both cases $s$ remains real in some vicinity of the roots $q_\pm$.

It must be noted that numerical trials did not yield a physically relevant combination of parameters yielding $P < 0$, so a negative sign for $P$ is a hypothetical option. Remember that condition (36) cannot be satisfied for $\gamma < \gamma_*$ and some interval of $\gamma$ slightly above $\gamma_*$. Since only the satisfaction of all conditions (36), (38) and (39*a*) entails instability onset, the inability to satisfy (39*a*) for some $\gamma < \gamma^\dagger$ is of little importance. The main interval of interest is $\gamma$ close to unity.

Once proven that there exists a root $\gamma^\dagger$, recall that $y(\gamma; \nabla\mathbf{f}, d)$ should actually have had two roots, both lying inside or outside the interval $0 < \gamma < 1$. This analysis cannot trace the second root because of the transcendentality of both $y$ and $c$. For the same reason the transition from roots situated within the interval $0 < \gamma < 1$ to roots outside it also cannot be traced.

### *3.5. Asymptotic analysis*

To complete the picture, an asymptotic analysis of the roots for short ($q \gg 1$) and long ($q \ll 1$) waves is performed. As explained before, at the latter limit no unstable modes are expected, and the analysis is performed to confirm that the anomaly does not affect the fundamental property of Turing instability initiation by waves of finite length.

A slight modification of (28*a*) is necessary due to the following reason. When $\gamma = 1, \lambda_1, \lambda_2$ are roots of (28*a*) only at $q = 0$; however, for $0 < \gamma < 1$ these are roots of order $1 - \gamma$ for all $q$, stable and of little interest. Rearranging (28*a*) and ignoring the terms $(s - \lambda_j)^{1-\gamma}$, $j \in \{1, 2\}$ yields

$$dq^4 - \frac{c_f}{2}(d-1)q^2\left((s-\lambda_2)^\gamma - (s-\lambda_1)^\gamma\right) + \frac{1}{2}(d+1)q^2\left((s-\lambda_2)^\gamma + (s-\lambda_1)^\gamma\right) + (s-\lambda_2)^\gamma (s-\lambda_1)^\gamma = 0, \qquad c_f = \frac{\nabla f_{11} - \nabla f_{22}}{\lambda_1 - \lambda_2}. \tag{59}$$

First, to explore the roots of (59) for $q \gg 1$, expand

$$s \sim w_1 q^{\nu_1}\left(1 + \sum_{j=2}^{\infty} w_j q^{\nu_j}\right), \qquad w_j \sim O(1) \quad \forall j \geqslant 1, \qquad \nu_j < 0, \qquad \nu_j < \nu_{j-1}\,\forall j \geqslant 2. \tag{60}$$

Existence of roots whose magnitude decays with $q$, i.e. $\nu_1 < 0$, is easily eliminated. Then the appropriate power is $\nu_1 = 2/\gamma$ and the leading order behaviour is obtained at $O(q^4)$

$$w_1^{2\gamma} + (d+1)w_1^\gamma + d = 0, \tag{61}$$

giving

$$w_1^\gamma = -1 \qquad \text{or} \qquad w_1^\gamma = -d. \tag{62}$$

Thus at the short wave range no real roots exist. Moreover, for $\gamma$ close to unity (strictly, $2/3 < \gamma < 1$) $\mathrm{Re}\, s < 0$ as

$$\arg w_1 = \frac{\pi}{\gamma} \in \left(\pi, \frac{3\pi}{2}\right). \tag{63}$$

To inspect the roots of (59) for $q \ll 1$, expand for one of the branches

$$s \sim \lambda_1 + q^{\nu_1} w_1 \left(1 + \sum_{j=2}^{\infty} q^{\nu_j} w_j\right), \qquad \nu_j > 0 \,\forall\, j. \tag{64}$$

Since at $q = 0$ the roots are complex conjugates and $\lambda_j \sim O(1)$, in close vicinity of $q = 0$ they will remain complex conjugates. By comparison of the resulting powers of $q$, $\nu_1 = 2/\gamma$ and at $O(q^2)$,

$$w_1^{\gamma} = -\frac{1}{2}(d+1) - \frac{\mathrm{i}}{2}(d-1)\frac{\nabla f_{11} - \nabla f_{22}}{\sqrt{4\det \nabla \mathbf{f} - \mathrm{tr}^2 \nabla \mathbf{f}}}. \tag{65}$$

Thus

$$\arg w_1 = \frac{\pi + \varphi}{\gamma}, \qquad \varphi = \arctan \frac{(d-1)(\nabla f_{11} - \nabla f_{22})}{(d+1)\sqrt{4\det \nabla \mathbf{f} - \mathrm{tr}^2 \nabla \mathbf{f}}}, \tag{66}$$

and

$$\mathrm{Re}\, s \sim \frac{1}{2}\mathrm{tr}\, \nabla \mathbf{f} + q^{2/\gamma} |w_1| \cos \frac{\pi + \varphi}{\gamma} + o(q^{2/\gamma}). \tag{67}$$

Different values of $\gamma$ will cause $\mathrm{Re}\, s$ to ascend or descend according to the sign of $\cos(\pi+\varphi)/\gamma$. When the anomaly exponents are close to unity ($1 - \gamma \ll 1$), the cosine is negative and $\mathrm{Re}\, s$ descends. The trend first changes at

$$\gamma = \frac{2}{3} + \frac{2\varphi}{3\pi} \tag{68}$$

and then again and so on at

$$\gamma = \frac{2}{2j+1} + \frac{2\varphi}{(2j+1)\pi}, \qquad j = 1, 2, \ldots . \tag{69}$$

It is possible to show that pure imaginary roots of (59) exist only at a long wave bifurcation point of the parameter manifold. Inserting $s = \mathrm{i}\omega$, $\omega \in \mathbb{R}$, the product of the roots $q_{\pm}^2$ is given by

$$q_+^2 q_-^2 = \frac{1}{d}(i\omega - \lambda_2)^{\gamma}(\mathrm{i}\omega - \lambda_1)^{\gamma}. \tag{70}$$

Bearing in mind that both $\omega$ and $q_{\pm}^2$ should be real, by complex conjugation of both sides of (70)

$$\omega_j = \mathrm{Im}\, \lambda_j, \qquad \mathrm{Re}\, \lambda_j = 0, \qquad j \in \{1, 2\} \tag{71}$$

and also

$$q_{\pm} = 0, \qquad \mathrm{tr}\, \nabla \mathbf{f} = 0, \qquad \omega = \pm 2\sqrt{\det \nabla \mathbf{f}}, \tag{72}$$

which, as expected, coincides with a Hopf bifurcation point for normal diffusion. As at the point $q = 0$ there is effectively no diffusion, its type (normal or anomalous) is immaterial.

To describe the range $q \sim O(1)$ another small parameter must be chosen. Let $1 - \gamma \ll 1$ and

$$s \sim s|_{\gamma=1} + \sum_{j=1}^{\infty} (\gamma - 1)^j w_j. \tag{73}$$

Then

$$(s - \lambda_j)^{\gamma} \sim (s - \lambda_j)[1 + (\gamma - 1)\log(s - \lambda_j) + O((1-\gamma)^2)]. \tag{74}$$

Substituting into (59) and collecting terms of order $O(1-\gamma)$ gives

$$w_1 = \left(2s|_{\gamma=1} + (d+1)q^2 - \operatorname{tr}\nabla\mathbf{f}\right)^{-1} \left\{ \frac{1}{2}(d-1)q^2 \frac{\nabla f_{11} - \nabla f_{22}}{\lambda_1 - \lambda_2} \left(L(\lambda_2) - L(\lambda_1)\right) \right.$$
$$\left. + (s|_{\gamma=1} - \lambda_1)L(\lambda_2) + (s|_{\gamma=1} - \lambda_2)L(\lambda_1) + \frac{1}{2}(d+1)q^2 \left(L(\lambda_2) + L(\lambda_1)\right) \right\}, \tag{75}$$

where

$$L(\lambda_j) = (s|_{\gamma=1} - \lambda_j)\log(s|_{\gamma=1} - \lambda_j), \qquad j = \{1, 2\}. \tag{76}$$

Hence the function $w_1$ is analytical at $q = 0$, and the bell-shaped curve $s(q^2)$, known for normal diffusion, is slightly shifted, yet does not undergo any essential changes. Since for $\gamma = 1$ the curve possesses a maximum within the range $(q_-^2, q_+^2)$ where $s \in \mathbb{R}$, for $1-\gamma \ll 1$ the situation should be similar. Differentiating (59) with respect to $q^2$ and seeking an extremum yields

$$2dq_{\text{ext}}^2 + \frac{1}{2}(d+1)q_{\text{ext}}^2\left((s_{\text{ext}} - \lambda_2)^\gamma + (s_{\text{ext}} - \lambda_1)^\gamma\right)$$
$$- \frac{c_f}{2}(d-1)q_{\text{ext}}^2\left((s_{\text{ext}} - \lambda_2)^\gamma - (s_{\text{ext}} - \lambda_1)^\gamma\right) = 0. \tag{77}$$

Using (59) again simplifies (77) to a tractable form

$$dq_{\text{ext}}^4 = (s_{\text{ext}} - \lambda_2)^\gamma (s_{\text{ext}} - \lambda_1)^\gamma, \tag{78}$$

whence

$$s_{\text{ext}} = \tfrac{1}{2}\left(\operatorname{tr}\nabla\mathbf{f} + \sqrt{\operatorname{tr}^2\nabla\mathbf{f} - 4\det\nabla\mathbf{f} + 4\left(dq_{\text{ext}}^4\right)^{1/\gamma}}\right). \tag{79}$$

The root with the plus sign was chosen because, for $\gamma = 1$, the extremum and the discriminant are positive. It is possible to rearrange (77) as

$$4 - (d-1)\frac{\nabla f_{11} - \nabla f_{22}}{\lambda_1 - \lambda_2}\left(\chi - \frac{1}{d\chi}\right) + (d+1)\left(\chi + \frac{1}{d\chi}\right) = 0 \tag{80}$$

with

$$\chi = \frac{q_{\text{ext}}^2}{(s_{\text{ext}} - \lambda_1)^\gamma} \tag{81}$$

and to solve the quadratics for $\chi$:

$$\chi = \frac{2}{d}\frac{\mathrm{i}d\sqrt{\Delta} \mp (d-1)\sqrt{d(-\nabla f_{12}\nabla f_{21})}}{(d-1)(\nabla f_{11} - \nabla f_{22}) - \mathrm{i}(d+1)\sqrt{\Delta}}, \qquad \Delta = 4\det\nabla\mathbf{f} - \operatorname{tr}^2\nabla\mathbf{f} > 0. \tag{82}$$

In order to choose the correct root, note that by definition (81)

$$s_{\text{ext}} - \lambda_1 = q_{\text{ext}}^{2/\gamma}\chi^{-1/\gamma} = \frac{1}{2}\sqrt{4\left(dq_{\text{ext}}^4\right)^{1/\gamma} - \Delta} - \frac{\mathrm{i}}{2}\sqrt{\Delta}$$
$$= d^{1/(2\gamma)}q_{\text{ext}}^{2/\gamma}\exp\left(-\mathrm{i}\arctan\sqrt{\frac{\Delta}{4\left(dq_{\text{ext}}^4\right)^{1/\gamma} - \Delta}}\right). \tag{83}$$

Also

$$|\chi|^2 = \frac{q_{\text{ext}}^4}{(s_{\text{ext}} - \lambda_1)^\gamma (s_{\text{ext}} - \lambda_2)^\gamma} = \frac{1}{d}. \tag{84}$$

Therefore the correct root should be such that $\arg\chi > 0$. Then

$$\arg\chi = \gamma \arctan\sqrt{\frac{\Delta}{4\left(dq_{\text{ext}}^4\right)^{1/\gamma} - \Delta}}, \tag{85}$$

giving

$$q_{\text{ext}}^2 = \frac{1}{\sqrt{d}}\left(\frac{\sqrt{\Delta}}{2}\csc\frac{\arg\chi}{\gamma}\right)^{\gamma} \tag{86}$$

and

$$s_{\text{ext}} = \frac{1}{2}\left(\operatorname{tr}\nabla\mathbf{f} + \sqrt{\Delta}\cot\frac{\arg\chi}{\gamma}\right). \tag{87}$$

Recalling the monotonic descent of the cotangent,

$$s_{\text{ext}}|_{\gamma=1} > s_{\text{ext}}|_{\gamma<1}, \tag{88}$$

i.e., the maximal growth rate is smaller in an anomalous system under the stated assumptions. It is possible to compute the value of $\gamma$ where instability disappears: $s_{\text{ext}} = 0$ at

$$\gamma = \frac{\arg\chi}{\arctan(\sqrt{\Delta}/(-\operatorname{tr}\nabla\mathbf{f}))} > 0. \tag{89}$$

## 4. Discussion

The investigation treated a two species, rotationally invariant sub-diffusive system with an integro-differential memory operator involving both reaction and diffusion processes. It was shown that in Laplace–Fourier space the terms are uncoupled similarly to normal diffusion and anomalous models where anomaly is represented by simple fractional derivatives.

For anomaly exponents close to unity the critical diffusion coefficients' ratio $d$ is higher than normal due to more severe conditions imposed on the system parameters. This property renders the anomalous system always less unstable than normal. The normal bell-shaped curve of the growth rate versus the wave number is shifted, so that the maximal growth rate is diminished as the anomaly exponent becomes smaller. Further reduction of $\gamma$ for fixed $d$ will eventually stabilize the system.

Asymptotic analysis showed that the dispersion relation possesses no real roots at the limit of short waves (large $q$). At the limit of long waves (small $q$) the growth rate curves coincide at the stable points $\lambda_1, \lambda_2$ for all $\gamma$; however, the normally positive slope changes sign intermittently for separated sets of $\gamma$.

In general, no pure imaginary roots exist. They are obtained formally at a long wave (Hopf) bifurcation point, where another stable solution should emerge.

Nonlinear instability theory is of interest for future research, in particular, addressing the essentially differing mechanisms of 'aging' of the reaction products and the properties of corresponding bifurcation points.

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## Appendix. Model with distinct waiting time distributions

The dispersion relation (10) can be obtained via another model where $\psi_j$ are not equal, but the jumps take place only to neighbour sites. Similarly to the model described in the main text, a small density perturbation $\mathbf{n}$ about a spatially uniform homogeneous state of a system of reagents evolves in accordance with the gradient $\nabla\mathbf{f}$ of the underlying nonlinear kinetic function $\mathbf{f}$. Following the ideas in [7], let $\mathbf{J}_i^-$ and $\mathbf{J}_i^+$ be the loss and gain flux vectors of particles $\mathbf{n}$ (without reaction) in the $i$th site of a one-dimensional lattice. Then the balance equation at this site is given by

$$\frac{\partial}{\partial t}\mathbf{n}_i = \mathbf{J}_i^+ - \mathbf{J}_i^- + \nabla\mathbf{f}\,\mathbf{n}_i. \tag{A.1}$$

In an unbiased random walk the gain and loss flux quantities are related to one another via neighbour sites:

$$\mathbf{J}_i^+ = \tfrac{1}{2}\mathbf{J}_{i-1}^- + \tfrac{1}{2}\mathbf{J}_{i+1}^-. \tag{A.2}$$

Combining (A.2) with (A.1),

$$\frac{\partial}{\partial t}\mathbf{n}_i = \frac{1}{2}\mathbf{J}_{i-1}^- + \frac{1}{2}\mathbf{J}_{i+1}^- - \mathbf{J}_i^- + \nabla\mathbf{f}\,\mathbf{n}_i. \tag{A.3}$$

Passing from discrete formulation to continuum and generally higher spatial dimensions,

$$\frac{\partial\mathbf{n}}{\partial t} = \nabla^2\mathbf{J}^- + \nabla\mathbf{f}\,\mathbf{n} \qquad \text{in} \quad \Omega. \tag{A.4}$$

As before, the domain $\Omega$ is the whole space or has a rectangular shape, and an initial condition $\mathbf{n}(\mathbf{r}, 0)$ is prescribed. The boundary conditions are taken either periodic or zero flux across the domain boundary.

The diffusion rate of each species in the system is governed by a waiting time distribution function, prescribing the probability to find a particle at every given moment $t$ located at $\mathbf{r}$. The particle either rested there from the beginning of the reaction or migrated there at time $t'$ from position $\mathbf{r}'$. Take $\psi$ to be a diagonal matrix of dimension $n \times n$ whose non-zero entries $\psi_{jj}$ are the waiting distribution functions of species $n_j$. It is assumed that the chemical reaction does not alter the 'age' of the particles. With the composition governed by linear kinetics (3), it is possible to express the connection between the loss flux vector at time $t$ and the gain flux at that site in the past:

$$\mathbf{J}_i^-(t) = \psi(t)\,\mathrm{e}^{\nabla\mathbf{f}t}\mathbf{n}_i(0) + \int_0^t \psi(t-t')\,\mathrm{e}^{\nabla\mathbf{f}(t-t')}\mathbf{J}_i^+(t')\,\mathrm{d}t'. \tag{A.5}$$

To paraphrase, to leave site $i$ at time $t$ a particle must have been there from the beginning or moved there $t - t'$ time ago. Using equation (A.1), replacing the discrete site $i$ by continuous position $\mathbf{r}$ and omitting the minus superscript,

$$\mathbf{J}(\mathbf{r}, t) = \psi(t)\,\mathrm{e}^{\nabla\mathbf{f}t}\mathbf{n}(\mathbf{r}, 0) + \int_0^t \psi(t-t')\,\mathrm{e}^{\nabla\mathbf{f}(t-t')}\left(\frac{\partial}{\partial t'}\mathbf{n}(\mathbf{r}, t') - \nabla\mathbf{f}\,\mathbf{n}(\mathbf{r}, t') + \mathbf{J}(\mathbf{r}, t')\right)\mathrm{d}t'. \tag{A.6}$$

For the particular case $n = 1$, $\nabla\mathbf{f} = -k$ equation (A.6) was derived in [7].

Upon writing the vectors $\mathbf{n}$ and $\mathbf{J}$ in a basis $\mathbf{v}_j$ of the eigenspace of $\nabla\mathbf{f}$ as

$$\mathbf{J} = \sum_{j=1}^{n}\mathcal{J}_j(\mathbf{r}, t)\mathbf{v}_j, \qquad \mathbf{n} = \sum_{j=1}^{n}\mathcal{N}_j(\mathbf{r}, t)\mathbf{v}_j, \tag{A.7}$$

temporal Laplace transform of (A.6) and consequent simplification via the convolution theorem and the argument shift (13) yield

$$\sum_{j=1}^{n} \tilde{\mathcal{J}}_j (I - \tilde{\psi}(s-\lambda_j))\mathbf{v}_j = \sum_{j=1}^{n} (s-\lambda_j)\tilde{\mathcal{N}}_j \tilde{\psi}(s-\lambda_j)\mathbf{v}_j, \tag{A.8}$$

wherein transformed quantities are denoted by tildes. For sub-diffusion of exponent $\gamma$ the waiting time distribution matrix

$$\begin{pmatrix} \psi_1(t) & & 0 \\ & \ddots & \\ 0 & & \psi_n(t) \end{pmatrix}$$

is known to have the following expansion in Laplace space:

$$\tilde{\psi}(s) \sim \begin{pmatrix} 1-(\tau_1 s)^\gamma & & 0 \\ & \ddots & \\ 0 & & 1-(\tau_n s)^\gamma \end{pmatrix} + o(|s|^\gamma), \tag{A.9}$$

where $\tau_1, \ldots, \tau_n$ are constant [7]. Then the leading order evolution equation becomes

$$\sum_{j=1}^{n} \tilde{\mathcal{J}}_j (s-\lambda_j)^\gamma C^{-1}\mathbf{v}_j = \sum_{j=1}^{n} (s-\lambda_j)\tilde{\mathcal{N}}_j \mathbf{v}_j \tag{A.10}$$

with

$$C^{-1} \stackrel{\text{def}}{=} \begin{pmatrix} \tau_1^\gamma & & 0 \\ & \ddots & \\ 0 & & \tau_n^\gamma \end{pmatrix}, \tag{A.11}$$

in matrix form reading

$$C^{-1} V (Is - \Lambda)^\gamma V^{-1} \tilde{\mathbf{J}} = (Is - \nabla\mathbf{f})\tilde{\mathbf{n}} \tag{A.12}$$

or

$$\tilde{\mathbf{J}} = (Is - \nabla\mathbf{f})^{-\gamma} C (Is - \nabla\mathbf{f})\tilde{\mathbf{n}}. \tag{A.13}$$

Applying the temporal Laplace transform and spatial Fourier transform (denoted by hat) to (A.4) gives

$$(Is - \nabla\mathbf{f} + q^2 (Is - \nabla\mathbf{f})^{-\gamma} C (Is - \nabla\mathbf{f}))\hat{\tilde{\mathbf{n}}} = \hat{\mathbf{n}}(\mathbf{q}, 0), \tag{A.14}$$

granting the dispersion relation to be

$$\det(Is - \nabla\mathbf{f} + q^2 (Is - \nabla\mathbf{f})^{-\gamma} C (Is - \nabla\mathbf{f})) = 0. \tag{A.15}$$

In the context of pattern formation the pertinent homogeneous state is stable in the absence of diffusion, i.e., waves of small wave number cannot manifest instability. With $q = 0$ the dispersion relation reduces to $\det(Is - \nabla\mathbf{f}) = 0$, and its roots are the stable eigenvalues of $\nabla\mathbf{f}$. Hence the matrix $Is - \nabla\mathbf{f}$ is non-singular and (A.15) becomes

$$\det((Is - \nabla\mathbf{f})^\gamma + q^2 C) = 0, \tag{A.16}$$

coinciding with (24). Here the terms of reaction and diffusion processes are uncoupled, a result difficult to foresee *a priori* from the model formulation due to the joint memory term.

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