Solution of weakly compressible isothermal flow in landfill gas collection networks

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Abstract. Pipe networks collecting gas in sanitary landfills operate under the regime of a weakly compressible isothermal flow of ideal gas. The effect of compressibility has been traditionally neglected in this application in favour of simplicity, thereby creating a conceptual incongruity between the flow equations and thermodynamic equation of state. Here the flow is solved by generalisation of the classic Darcy-Weisbach equation for an incompressible steady flow in a pipe to an ordinary differential equation, permitting continuous variation of density, viscosity and related fluid parameters, as well as head loss or gain due to gravity, in isothermal flow. The differential equation is solved analytically in the case of ideal gas for a single edge in the network. Thereafter the solution is used in an algorithm developed to construct the flow equations automatically for a network characterised by an incidence matrix, and determine pressure distribution, flow rates and all associated parameters therein.

Keywords : flow rate, head loss, Darcy-Weisbach coefficient, pipe network, weakly compressible flow

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1. Background

1.1. The landfill problem

Landfill collection networks manifest flow conditions unique both within the gas industry and the field of hydraulics. Most industrial applications involving gas pipeline flow are characterised by a fully compressible regime, whilst hydraulic networks usually conduct incompressible fluid, whence the variation of temperature is of scarce interest. The flow in a landfill network falls somewhere in between: the fluid is weakly compressible and the flow cannot be solved without determination of the temperature variable. Moreover, this particular type of network presents the additional complication of multiple source points with streams of distinct chemical composition mixing at each node. Traditionally this flow was approximated as incompressible for simplicity, albeit the mainstay complexity did not stem from the choice of a head loss model. The current contribution sets forth the theory that is conceptually consistent from the vantage point of fluid mechanics and thermodynamics, and accounts for properties peculiar to this application, such as differences in elevation between the network nodes and early onset of turbulence.

1.2. Classic Darcy-Weisbach equation

Historically Darcy-Weisbach equation was developed for steady incompressible flow upon the underpinning of dimensional analysis combined with empirical measurements to estimate the head loss $|\Delta p|$ in a horizontal pipe of diameter d and length ℓ:

$$|\Delta p| = \frac{1}{2}\rho u^2 f \frac{\ell}{d}, \tag{1}$$

wherein ρ denotes the fluid density, u is flow velocity, and f represents a dimensionless friction coefficient as given by the Colebrook equation (Colebrook, 1939; White, 1999) for a fully turbulent regime

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right) \tag{2}$$

and involving the relative surface roughness ε and Reynolds number Re, both based on diameter d. Some remarks regarding (2) are in order. The salient moment of this phenomenological approach is a realistic estimate of $|\Delta p|$ owing to the inclusion of interaction with a non-smooth surface through f, whilst Navier-Stokes flow equations entail solely fluid properties. That f is a function of Re and ε can be obtained by dimensional analysis, cf. Benedict (1980). Full turbulence in the current problem of landfill networks is obtained somewhat earlier than the commonly concomitant with (2) threshold of Re > 3000 (Moody, 1944), the reason being twofold: one, Moody's diagram is for flow in a single pipe with no geometric disturbances, whilst in the network the pipes' junctures engender turbulence; and two, the fluid comes into the network from vertical wells, wherein flow rate measuring devices are installed, e.g. orifice plates, rendering the flow turbulent even at very low Re numbers. Characteristic Re numbers

in this application are $10^3 - 10^5$. Equation (2) is implicit in $\sqrt{f}$ and must be solved numerically or equivalently through Lambert W function (Corless et al., 1996).

Forthright application to flows with inherent compressibility is impossible, however. The current contribution is inspired by the unique flow regime occurring in gas collection networks in landfills. Simplistically depicted, the landfill comprises an underground organic mass, wherein anaerobic biodegradation takes place; a covering substratum with an imbedded system of collection pipes; and an extraction facility. The biodegradation of waste is exothermal, and the amount of gas generated is rarely uniform within the landfill. The gas is conducted through a series of wells into the system of pipes, the flow induced by suction created at the extraction facility. The landfill is not absolutely impermeable to ambient air, but when the landfill pressure is properly monitored, there is no exchange between the landfill and atmosphere. Two circumstances decide the thermal regime: one, the pipe network is situated within the substratum sufficiently far both from the landfill cavity, where the chemical reactions occur, and from the surface; and two, the pipes are not insulated, allowing dissipation of friction generated heat. A thermal equilibrium thus ensues within the substratum, midmost layers thereof maintaining a constant temperature, virtually unaffected by seasonal changes. Obviously, where the pipes are above ground, isothermal conditions are unlikely to occur. Henceforth the flow within the collection network is taken isothermal. The thermodynamic properties of the fluid render the assumption of ideal gas tenable, forthwith imparting flow compressibility, as is seen from the state equation

$$p = \rho R T, \tag{3}$$

p denoting pressure, T – temperature and R – the gas constant, whereby pressure p and density ρ must perforce vary proportionately since T and R are fixed.

Estimation of flow rates in the collection network is a basic step in landfill design, whence a relation akin to (1) is of the essence. Surprisingly no adequate model is in use. The sundry branches of the gas industry have developed empiric formulae galore, each suitable to a particular flow regime and swiftly losing accuracy when applied elsewhere (Menon, 2005). Some justification of the physical kernel of the aforesaid formulae can be found in Coelho and Pinho (2007) and references therein. In addition to custom fit exponents two features of import prevail: compressibility factor $z < 1$ and efficiency η, or variants thereof. Introduction of the compressibility factor is equivalent to adoption of a more general equation of state

$$p = z \rho R T, \qquad z \leqslant 1, \tag{4}$$

however here $z < 1$ will be incongruent with the regime of weak compressibility. The multiplicative efficiency factor η is a crude attempt to account for natural phenomena pertaining to head loss that elude properly grounded, yet pragmatic modelling. By contrast, Darcy friction factor f manifests noteworthy generality through its continuous dependence on physical parameters of both fluid and conduit, namely ε and Re.

Changes in fluid density in the landfill collection network rarely are such as to render the accuracy of (1) with a fixed value of ρ appalling. Characteristic Mach numbers range below 0.1, therefore the flow is only weakly compressible. Nonetheless, application of (1) is conceptually incorrect and entails significant error when pipe lengths are large. In that light the current contribution generalises (1) for continuously varying density ρ, yet preserves the valuable complexion of Darcy friction factor f. Affording accurate estimates of head loss in landfill pipework is essential for determination of gas quantities extracted, as will become apparent infra, and subsequent effectuation of sustainable practices such as diversion of the gas for beneficial uses.

2. Continuous Darcy-Weisbach equation

Hereinafter integral mass and momentum equations are used to derive an ordinary differential equation to supplant (1). The layout of landfill pipes follows local landscape, therefore inclusion of gravity in the balance of momentum is imperative, as under low flow conditions head loss or gain due to sloping terrain becomes non-negligible. Consider a control volume of a straight pipe of length ℓ and radius r, inclined at angle α as in figure 1. The flow is steady. Mass conservation equation reads (Batchelor, 1990)

$$\oint_{\partial V} \rho \mathbf{u} \cdot d\mathbf{s} = 0, \tag{5}$$

wherein ∂V is the surface of the control volume V, ρ is fluid density, $\mathbf{u}$ denotes velocity vector and $\mathbf{s}$ – area vector with the normal directed outwards. Integration yields

$$\rho_{\text{in}} u_{\text{in}} = \rho_{\text{out}} u_{\text{out}} \tag{6}$$

with subscripts $(\,\cdot\,)_{\text{in}}$ and $(\,\cdot\,)_{\text{out}}$ referring to the respective cross-sections, and $u = |\mathbf{u}|$ being the velocity magnitude. For consistence hereunder velocity and other quantities inherently varying within the cross-section tacitly refer to equivalent uniform values. Momentum conservation equation reads (Batchelor, 1990)

$$\int_{\partial V} \mathbf{u} \left(\rho \mathbf{u} \cdot d\mathbf{s} \right) + \int_{\partial V} p \, d\mathbf{s} = \int_{V} \rho \, \mathbf{f}_{\text{body}} \, dV + \mathbf{f}_{\text{surf}}, \tag{7}$$

wherein p is fluid pressure, $\mathbf{f}_{\text{body}}$ is the gravity vector, and the only surface force $\mathbf{f}_{\text{surf}}$ is due to the shear stress τ within the boundary layer, via dimensional analysis given by a friction coefficient c_f and integration of dynamic pressure $\frac{1}{2}\rho u^2$ along the cylinder walls:

$$\left| \mathbf{f}_{\text{surf}} \right| = \pi r \int_{0}^{\ell} c_f \rho u^2 dx. \tag{8}$$

Applying (7) in the axial direction yields

$$-\rho_{\text{in}} u_{\text{in}}^2 + \rho_{\text{out}} u_{\text{out}}^2 - p_{\text{in}} + p_{\text{out}} = \frac{g \sin \alpha}{\pi r^2} \int_{V} \rho \, dV - \frac{1}{r} \int_{0}^{\ell} c_f \rho u^2 dx. \tag{9}$$
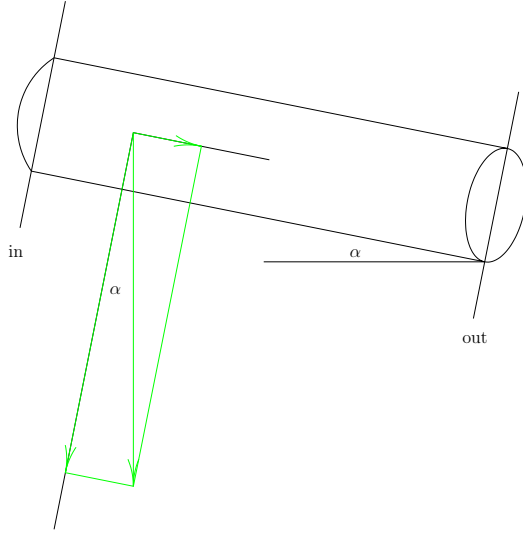
Figure 1: Control volume schematic. Angle α is defined positive as shown, conforming to head gain due to gravity.

For incompressible flow, to wit density constant throughout $\rho_{\text{in}} = \rho_{\text{out}} \overset{\text{def}}{=} \rho$ and implying velocity similarly constant $u_{\text{in}} = u_{\text{out}} \overset{\text{def}}{=} u$, equation (9) becomes

$$p_{\text{out}} - p_{\text{in}} = \left(g\ell \sin\alpha - \frac{u^2}{r} \int_0^\ell c_f dx \right)\rho, \tag{10a}$$

recovering a gravity inclusive variant of (1)

$$p_{\text{out}} - p_{\text{in}} = \left(g \sin\alpha - \frac{1}{2} u^2 \frac{f}{d} \right)\rho\ell \tag{10b}$$

upon setting $d = 2r$ and

$$\frac{1}{\ell} \int_0^\ell c_f dx = \frac{1}{4} f. \tag{10c}$$

In this sense the classic Darcy-Weisbach friction coefficient f is a multiple of the average friction coefficient, or inversely, thenceforth the friction coefficient is regarded as an effective average value that up to a multiplicative constant locally equals the Darcy-Weisbach coefficient f.

To continue, suppose a generic equation of state, incorporating both (3) and (4) as particular cases (even if $z < 1$, the thermodynamic properties of the fluid are not expected to vary insofar as to render z a function of x),

$$\rho = \mathfrak{F}(p, T), \tag{11}$$

for isothermal flow giving together with (6)

$$\frac{u_{\text{out}}}{u_{\text{in}}} = \frac{\rho_{\text{in}}}{\rho_{\text{out}}} = \frac{\mathfrak{F}(p_{\text{in}}, T)}{\mathfrak{F}(p_{\text{out}}, T)}. \tag{12}$$

Further combining with (6) and (9) gives

$$1 - \frac{\mathfrak{F}(p_{\text{in}}, T)}{\mathfrak{F}(p_{\text{out}}, T)} - \frac{p_{\text{out}} - p_{\text{in}}}{\rho_{\text{in}} u_{\text{in}}^2} = \frac{1}{\rho_{\text{in}} u_{\text{in}}^2} \left(\frac{1}{r} \int_0^\ell c_f \rho u^2 dx - \frac{g \sin \alpha}{\pi r^2} \int_V \rho \, dV \right). \tag{13}$$

To facilitate a continuous variation of ρ along the control volume, map the finite length ℓ to infinitesimally small $\delta\ell$, whereby $\rho_{\text{in}} \longmapsto \rho$, $u_{\text{in}} \longmapsto u$, $p_{\text{in}} \longmapsto p$, $p_{\text{out}} \longmapsto p + \delta p$, $\ell \longmapsto \delta\ell$ in (10c) and

$$\mathfrak{F}(p_{\text{out}}, T) \sim \mathfrak{F}(p_{\text{in}}, T) + \left. \frac{d\mathfrak{F}}{dp} \right|_{\text{in}} \delta p + \mathcal{O}\left(\delta p^2 \right). \tag{14}$$

Then (13) becomes

$$\delta p \left(\frac{1}{\mathfrak{F}} \frac{d\mathfrak{F}}{dp} - \frac{1}{\rho u^2} \right) = \left(\frac{f}{4r} - \frac{g \sin \alpha}{u^2} \right) \delta\ell \tag{15}$$

upon completion of integrals and omission of higher order terms. At the limit the following ordinary differential equation ensues

$$\lim_{\substack{\delta p \longrightarrow 0 \\ \delta\ell \longrightarrow 0}} \frac{\delta p}{\delta\ell} = \frac{dp}{d\ell} = \left(\frac{f}{4r} - \frac{g \sin \alpha}{u^2} \right) \Big/ \left(\frac{1}{\mathfrak{F}} \frac{d\mathfrak{F}}{dp} - \frac{1}{\rho u^2} \right). \tag{16}$$

Note that (16) was derived through integral mass and momentum conservation with a generic equation of state and without making any assumptions regarding the variation of the friction factor f. A less generic version can be found in classic literature, derived for an ideal gas, horizontal flow ($\alpha = 0$) and f gratuitously assumed constant throughout (Benedict, 1983; Ward-Smith, 1980). The friction factor f depends on the Reynolds number, thus introducing a dependence on the fluid viscosity, which is known to vary with temperature. Hence implicitly (16) involves all properties sufficient to determine the thermodynamic state of a substance and will require a specific equation of state for solution.

The function $\mathfrak{F}$ is readily identified as $\mathfrak{F} = p/(zRT)$, and (16) simplifies to

$$\frac{dp}{d\ell} = \left(\frac{f}{4r} - \frac{g \sin \alpha}{u^2} \right) \Big/ \left(\frac{1}{p} - \frac{1}{\rho u^2} \right). \tag{17}$$

If the equivalence (10c) is premised, i.e. the constraint of Darcy-Weisbach relation conforming solely to incompressible flow is relaxed ad hoc, equation (17) can be obtained from the differential form of the mass and momentum conservation equations:

$$d(\rho u) = 0 \tag{18a}$$

$$u \, du + \frac{dp}{\rho} - g \sin \alpha \, d\ell + \frac{f}{4r} u^2 d\ell = 0. \tag{18b}$$

Before integration of (17) is attempted, it is necessary to determine how the friction coefficient f varies. By (2) f is a function of ε, fixed for a given pipe, and Re, with the expression of mass flow rate

$$\dot{m} = \pi r^2 \rho u, \tag{19}$$

to be written as

$$\mathrm{Re} = \frac{4\dot{m}}{\pi \mu d}. \tag{20}$$

The mass rate $\dot{m}$ is constant by (6). The viscosity μ depends on temperature.§ Since the flow is isothermal, μ is constant, whereby so are Re and f.

Recollecting (3) and using (19), for $\alpha \neq 0$ equation (17) is forthwith cast into the more elegant form

$$\frac{dp}{d\ell} = \frac{g \sin \alpha}{RT} \frac{p(p^2 - p_{\mathrm{eq}}^2)}{p^2 - p_\infty^2}, \tag{21a}$$

wherein

$$p_{\mathrm{eq}}^2 = \left(\frac{\dot{m}RT}{2\pi r^2}\right)^2 \frac{f}{gr \sin \alpha}, \quad p_\infty = \frac{\dot{m}\sqrt{RT}}{\pi r^2}. \tag{21b}$$

For $\alpha > 0$, i.e. downward slope, the pressure p_{eq} entails a perfect balance of head gain due to gravity and loss due to friction. For $\alpha < 0$, i.e. upward slope, (21) possesses no fixed points as $p_{\mathrm{eq}}^2 < 0$. The singular point p_∞ corresponds to the subsonic velocity $u = \sqrt{RT}$ ‖ and has been obtained heretofore for horizontal geometry (Benedict, 1983). In the current system this singular point is never approached, the flow being weakly compressible with Mach number below 0.1 and $1.2 < \gamma < 1.25$.

2.1. Solutions

Both forms (17) and (21) are separable and easily integrated. A simple familiar case is that of $\dot{m} = 0$, whereby the integration is immediate, recovering the result for hydrostatic pressure change

$$\ln \frac{p_{\mathrm{out}}}{p_{\mathrm{in}}} = \frac{g\ell \sin \alpha}{RT}. \tag{22}$$

For $\alpha = 0$, i.e. horizontal flow, integration of (17) over a pipe of length ℓ gives

$$\frac{4r}{f} \ln \frac{p_{\mathrm{out}}}{p_{\mathrm{in}}} - \frac{2\pi^2 r^5}{f\dot{m}^2 RT}\left(p_{\mathrm{out}}^2 - p_{\mathrm{in}}^2\right) = \ell, \tag{23a}$$

§ Common models are Sutherland's formula $\dfrac{\mu}{\mu_o} = \left(\dfrac{T}{T_o}\right)^{3/2} \dfrac{T_o + s}{T + s}$ (Crane, 1982) or exponential law $\dfrac{\mu}{\mu_o} = \left(\dfrac{T}{T_o}\right)^n$ (White, 1999) with μ_o, T_o, s, n experimentally determined constants.

‖ For an adiabatic flow of ideal gas Mach number of unity is reached at $u = \sqrt{\gamma RT}$ with $\gamma > 1$ being the ratio of heat capacities c_p/c_v.

a transcendental equation that cannot be solved as is for an explicit expression akin to (10). For $\alpha \neq 0$ form (21) is preferable and upon integration yields ¶

$$\frac{p_\infty^2}{p_{eq}^2} \ln \frac{p_{out}}{p_{in}} + \frac{1}{2}\left(1 - \frac{p_\infty^2}{p_{eq}^2}\right) \ln \frac{p_{out}^2 - p_{eq}^2}{p_{in}^2 - p_{eq}^2} = \frac{g\ell \sin \alpha}{RT}, \qquad (23b)$$

similarly insoluble for p_{out} in closed form without further comparison of the various quantities' magnitude. Equations (23) pose no difficulty when solved numerically by methods commonly used for non-linear equations.

To exemplify the differences effected by the continuous variation of density, equations (23) and (10) were solved for two situations:

† downstream prediction: p_{in} given, compute p_{out} $\left(\dfrac{p_{out}}{p_{in}} < 1 \text{ for horizontal flow}\right)$

† upstream prediction: p_{out} given, compute p_{in} $\left(\dfrac{p_{in}}{p_{out}} > 1 \text{ for horizontal flow}\right)$.

Albeit the former is the more commonly encountered case, the upstream prediction is the inherently applicable case for the landfill networks, since the pressure maintained at the extraction facility dictates the conditions up the stream. The magnitude of the difference in head loss as predicted by the continuous model (23) versus the classic (10) varies with the flow parameters. For a typical example see figure 2. The conceptual disparity between (23) and (10) is summarily discernible: by (10) the density is constant and pressure decreases linearly down the stream, whereas (23) imbeds a continuous change of density through (3) and entails a non-linear diminution of pressure. When the density is deemed constant and taken as the entry value, (10) overestimates the pressure downstream (figure 2, left panel). Similarly, with an exit value (10) will overestimate the pressure up the stream (figure 2, right panel). As expected, for small ℓ density changes scarcely evolve, rendering the foregoing difference negligible. Over long distances the changes due to compressibility are more pronounced. The continuous model (23) should therefore be considered the correct one in the case of weakly compressible flow, where usage of (10) is conceptually erroneous, albeit as a discrete approximation giving a close estimate for sufficiently small ℓ.

3. Application to landfill gas collection network

The continuous Darcy-Weisbach equation (23) was used to solve the flow in a landfill collection network. The foremost property that distinguishes this network from a common hydraulic network is that each node serves as a source point of fluid with distinct thermodynamic properties. The gas in each well flowing from the landfill cavity upwards has its own composition that depends on the decay processes in that vicinity. The stream of each well mixes with the gas incoming into the node by pipes collecting gas from other parts of the network.

¶ The equilibrium solution $p_{in} = p_{out} = p_{eq}$ for $\alpha > 0$ does not require integration.
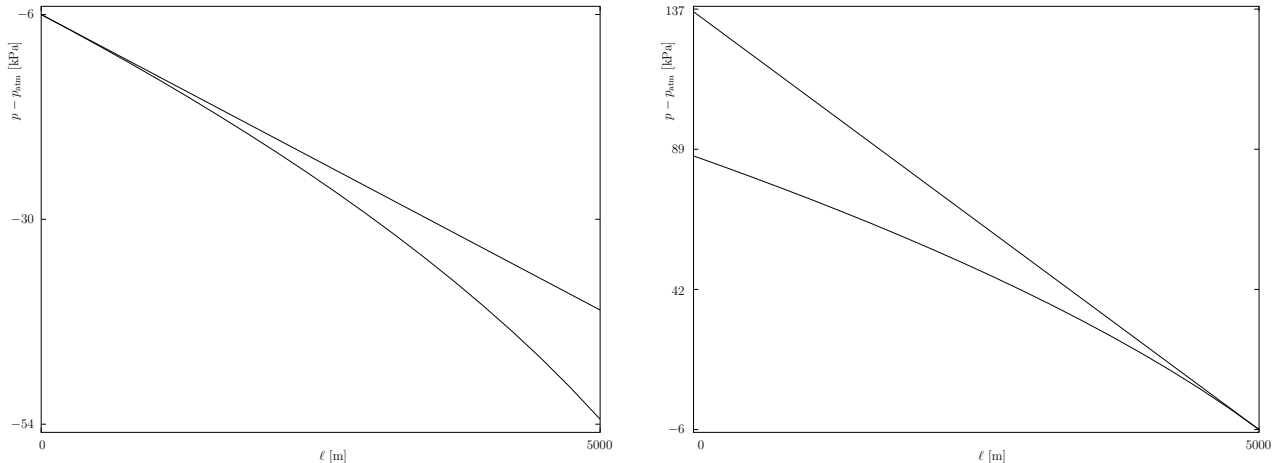
Figure 2: Comparison of head loss prediction with correct inclusion of compressibility effect (equation (23), thick black curve) versus classic Darcy-Weisbach formula neglecting density changes (equation (10), thin grey line). Left panel: downstream prediction (p_{in} given, compute p_{out}) results in overestimation of outlet pressure by (10) based on fixed entry density. Right panel: upstream prediction (p_{out} given, compute p_{in}) results in overestimation of inlet pressure by (10) based on fixed exit density. Parameters used: relative roughness $\varepsilon = 10^{-5}$ (absolute roughness of 1.5×10^{-6}m for plastic and drawn tubing, consult Crane (1982, A-23), and diameter as below), $T = 13^o$C, molar fractions $\left\{ x_{\text{CH}_4}, x_{\text{O}_2}, x_{\text{CO}_2}, x_{\text{N}_2} \right\} = \left\{ 0.5, 0.01, 0.45, 0.04 \right\}$, $\alpha = \arctan(0.05)$, $\dot{m} = 0.2$kg/s, $d = 0.15$m, $\mu = 1.4 \times 10^{-5}$Pa $\cdot$ s (corresponding Re $= 120,000$), reference pressure $p = 6$kPa below atmospheric (p_{in} for left panel and p_{out} for right panel). All numeric values given up to two significant digits.

A generic solution to such a system is conceptually impossible without a given network configuration. Hereunder the problem is posed for a general network with the aid of fundamentals of graph theory, and a solution algorithm is developed. Thereafter a solution is presented for a specific example of a network of sufficient difficulty to be representative with the purpose to establish that implementation of (23) was computationally feasible on par with the classic model (10).

The generic geometry of the system is as follows. The flow network is a graph with $\mathfrak{n}$ nodes corresponding to extraction wells and $\mathfrak{e}$ edges representing collection pipes. An example of typical complexity is given in figure 3. One of the nodes is designated as the flare, where sub-atmospheric pressure is imposed, inducing flow from the wells that act as sources for the network. Two parameters control the collection: suction strength at the flare as dictated by considerations of efficiency, and a maximal permissible pressure (sub-atmospheric) throughout the system as set by environmental and safety regulations[+] (Conestoga-Rovers, 2010). The well flow rates are deemed known input.[*]

To determine the flow rates in the collection pipes one is required to solve a system

[+] Excessive pressure bespeaks insufficient gas extraction and is bound to beget leaks through the landfill cavity boundary into the atmosphere.

[*] During landfill design phase well flow is estimated based on the amount and nature of refuse to be buried at that locality. When fully operational, each well's flow rate is measured. Certain field conditions necessitate additional care to achieve adequate accuracy (Nec and Huculak, 2015).
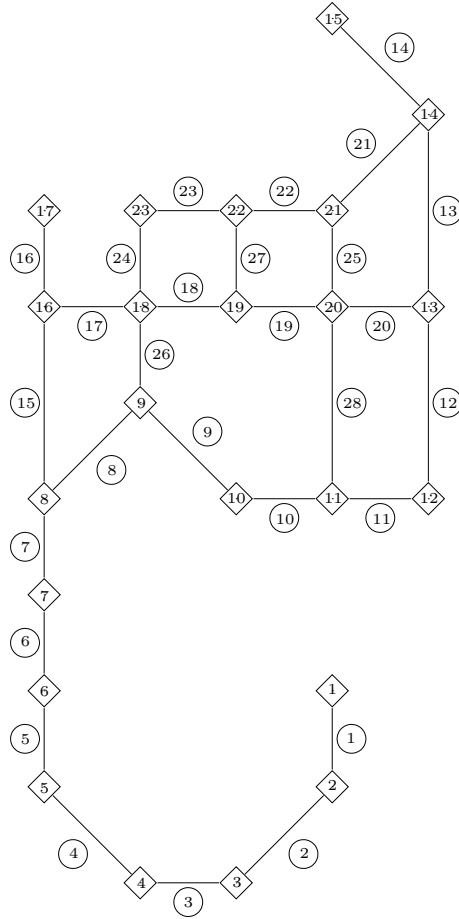
Figure 3: Example graph: vertices correspond to wells (numbered in diamonds), edges correspond to pipes (numbered in circles). Flare is located at node #1, where sub-atmospheric pressure is imposed. The general collection progress is from the remote wells toward the flare.

of non-linear equations. A network flow is governed by two principles: conservation of mass at the nodes and continuity of pressure over any closed path. A graph of $\mathfrak{n}$ nodes and $\mathfrak{e}$ edges contains $\mathfrak{e} - \mathfrak{n} + 1$ fundamental cycles (Godsil and Royle, 2001), i.e. number of independent closed paths, conforming to the number of pressure continuity equations required. Perforce the number of independent mass conservation equations will be $\mathfrak{n} - 1$. Hence define $\mathfrak{J}_{\text{in}}^i$ and $\mathfrak{J}_{\text{out}}^i$ respectively as the sets of incoming and outgoing pipes at the ith node. With $\dot{m}_j$ denoting the flow rate through the jth pipe, mass conservation equations read

$$\sum_{j \in \mathfrak{J}_{\text{in}}^i} \dot{m}_j + \dot{m}^{(i)} = \sum_{j \in \mathfrak{J}_{\text{out}}^i} \dot{m}_j, \quad i = 1, \ldots, \mathfrak{n} - 1, \tag{24a}$$

wherein $\dot{m}^{(i)}$ is the mass flow rate input by the ith well. Construct the network's incidence matrix as follows (Godsil and Royle, 2001). The matrix is of size $\mathfrak{n} \times \mathfrak{e}$. Each edge is assigned an arbitrary flow direction. In each column, corresponding to an edge, entry ij is set to equal (-1) if the flow in pipe j proceeds from node i, unity if the flow enters node i, and zero otherwise. Each row is then used to determine the set $\mathfrak{J}_{\text{in}}^i \cup \mathfrak{J}_{\text{out}}^i$ and

write (24a) in matrix form.

Define $\mathfrak{J}_c^k$ as the set of pipes forming the kth fundamental cycle. With Δp_j being the pressure drop over the jth pipe (sign determined by flow direction), pressure continuity equations read

$$\sum_{j \in \mathfrak{J}_c^k} \Delta p_j = 0, \quad k = 1, \ldots, \mathfrak{e} - \mathfrak{n} + 1. \tag{24b}$$

The fundamental cycles are obtained by finding the null space of the incidence matrix (Godsil and Royle, 2001). The dimension of that space gives the number of required pressure continuity equations. The vectors spanning the space allow to write (24b) in matrix form. Together (24a) and (24b) give a system of $\mathfrak{e}$ equations in $\mathfrak{e}$ unknown mass flow rates $\left\{ \dot{m}_j \right\}_{j=1}^{j=\mathfrak{e}}$, with (24a) being linear, whereas (24b) are non-linear by (23). It is possible to construct the incidence matrix conforming to an acyclic sub-graph of the network, sized $\mathfrak{n} \times (\mathfrak{n} - 1)$, such that the inverse of a square block $(\mathfrak{n} - 1) \times (\mathfrak{n} - 1)$ thereof, computed only once, enables the determination of $\mathfrak{n} - 1$ out of $\mathfrak{e}$ unknowns via (24a). Thence only the non-linear system (24b) in $\mathfrak{e} - \mathfrak{n} + 1$ unknowns requires iterative solution.

Solution of (24) is somewhat convoluted for the particular application of a landfill network, because the gas differs in composition between the different wells. Simple causality implies that properties of the mixture can be computed only in the downstream direction, whereas head loss must be computed starting with the prescribed flare pressure and proceeding upstream. Note that, identically to (10), both (23a) and (23b) accord the determination of $p_{\text{out}}(p_{\text{in}})$ (downstream prediction) and $p_{\text{in}}(p_{\text{out}})$ (upstream prediction) with equal ease. This attribute proves of import in the solution of (24b), since in a network of that ilk well flow rates might fluctuate significantly over prolonged periods of time as a cause of natural changes in chemical reactions within the landfill, bringing about flow reversal in particular parts of the network, an occurrence all the more likely when the terrain manifests elevation changes. In light of the above, when equation (23) supplants (10) in the computation of pressure, the impact on the overall computational complexity of (24) is peripheral.

3.1. Mixing

A further peculiarity of the landfill network is that each node is a conflux of streams of different chemical compositions. With the depth of description accorded by the integral mass and momentum conservation alone it is consistent to model the mixing as instantaneous. The prevalent components are CH_4, O_2, CO_2, N_2. The molar fractions of the mixture input by each well are measured and recorded as a quadruple $\left\{ x_{CH_4}, x_{O_2}, x_{CO_2}, x_{N_2} \right\}$. By definition

$$x_{CH_4} + x_{O_2} + x_{CO_2} + x_{N_2} = 1. \tag{25}$$

Upon completion of mixing at the ith node the molar fraction $x_{\mathfrak{Y}}$ of component $\mathfrak{Y}$ of the mixture entering the set of outgoing pipes $\mathfrak{J}^i_{\text{out}}$ is given by

$$x_{\mathfrak{Y}} = \left(\sum_{j\in\mathfrak{J}^i_{\text{in}}} \dot{m}_j x_{\mathfrak{Y}_j} + \dot{m}^{(i)} x_{\mathfrak{Y}}^{(i)} \right) \Big/ \left(\sum_{j\in\mathfrak{J}^i_{\text{in}}} \dot{m}_j + \dot{m}^{(i)} \right), \qquad (26)$$

wherein $x_{\mathfrak{Y}_j}$ and $x_{\mathfrak{Y}}^{(i)}$ are the fractions in j^{th} pipe and i^{th} well respectively. The verification that the newly obtained fractions sum to unity is immediate. Naturally the computation of gas composition throughout the network is causal and must proceed from the remotest well towards the flare. Given that the solution to the non-linear system (24) is perforce iterative, the mixing is ineluctably performed at every iteration. The gas constant R is then computed for each pipe

$$R_j = R_o \Big/ \sum_{\mathfrak{Y}} x_{\mathfrak{Y}} M_{\mathfrak{Y}}, \quad j = 1, \ldots, \mathfrak{e} \qquad (27)$$

with R_o being the universal gas constant, $\mathfrak{Y}$ running over all components of the mixture and $M_{\mathfrak{Y}}$ denoting the corresponding molar weights.

3.2. Pressure distribution

The head loss across the network must be similarly computed at each iteration of the numerical solution of (24). Starting at the flare, where a desired (design phase) or actual (operational landfill) sub-atmospheric pressure is prescribed, the pressure is computed sequentially upstream across the network with (23). Upon convergence the solution is examined for compliance with the safety criteria. If the landfill is still under design, the analysis determines where in the system the margin to maximal permissible pressure is too small. Pipe geometry is consequently to be adjusted (larger diameter or additional edges) until admissible margin is obtained. If the landfill is operational, when the pressure limit is approached at any point in the network, the operator will adjust the flow to the relevant wellheads, or if the problem appears to be of a more global nature, modify the suction strength at the flare.

3.3. Example network

System (24) was solved with the underlying geometry of the network in figure 3. The example features all properties pertinent to a landfill, rendering its generality sufficient: (a) substantial number of nodes $\mathfrak{n} = 23$, edges $\mathfrak{e} = 28$ and fundamental cycles $\mathfrak{e} - \mathfrak{n} + 1 = 6$; (b) exitence of isolated wells, namely $\mathfrak{J}^i_{\text{in}} = \emptyset$ for $i = 15, 17$; (c) a node designated as flare, $\mathfrak{J}^1_{\text{out}} = \emptyset$; (d) existence of crucial edges ($j = 8, 15$), whose contingent disconnexion entails flow reversal in several parts of the network and possible violation of the safety constraint delimiting the pressure to a maximal value throughout.

The following nominal real network geometry and landfill parameters were used: $80\text{m} \leqslant \{\ell_j\}^{\mathfrak{e}}_{j=1} \leqslant 450\text{m}$, $0.15\text{m} \leqslant \{d_j\}^{\mathfrak{e}}_{j=1} \leqslant 0.2\text{m}$, individual well volumetric flow

up to $0.04 \mathrm{m}^3/\mathrm{s}$, and gas composition with $0.45 \leqslant x_{\mathrm{CH_4}}^{(i)} \leqslant 0.55$, $0 \leqslant x_{\mathrm{O_2}}^{(i)} \leqslant 0.02$, $0.3 \leqslant x_{\mathrm{CO_2}}^{(i)} \leqslant 0.4$, the fraction $x_{\mathrm{N_2}}^{(i)}$ in accord with (25), $i = 1, \ldots, \mathfrak{n}$. The suction at the flare was set to $6\mathrm{kPa} < p_{\mathrm{flare}} < 25\mathrm{kPa}$ below atmospheric pressure, a feasible range for the landfill in question. One should bear in mind that the solution network is a model of the real system, thus the dimensions do not directly conform to the field installation. For instance, the pipe lengths are effective values adjusted to account for losses due to various valves, fittings, condensate drain devices etc. Moreover, each node might represent a cluster of wells. Being an unalienable part of landfill flow computation, such equivalences, extraneous to the specific head loss equation employed, are based on experience of the design and construction engineers. Hence the graph layout ought to be regarded as a framework, where numerous real pipework detail has been absorbed into effective dimensions. For this reason all values are given in precision of no more than two significant digits.

With the foregoing base parameters an increase in the following quantities was tested: (a) pipe length; (b) elevation difference between nodes (equivalent to increase in incline angle $|\alpha|$); (c) well input flow rate. It was found that formula (10) consistently led to overestimation of pressure at the nodes. For clarity of exposition the effect is illustrated for the higher values of extraction vacuum at the flare.

Figure 4 shows the variation of pressure within the system relatively to the flare along a path from the flare at node 1, corresponding to the lowest pressure, to well 15, the remotest point and highest pressure in the network. Discernible changes of slope occur at wells with significant mass input. The full potential of extraction is achieved when the maximal pressure is only slightly below atmospheric, i.e. the fluid at the remote nodes recovers most of the vacuum imposed at the flare. As exemplified below, this is where the use of (23) instead of (10) can make a difference, mitigating the conservative prediction by the latter approach.

The solution of (24) with the head loss correctly given by (23) was compared with the case, where (23) was replaced by the discrete approximation (10), the implementation thereof based on the fluid density at the exit of the pipe and kept fixed (for the sake of comparison only, in congruence with (10), albeit erroneously in the current application). Figure 5 shows the resulting overestimation of pressure by the traditional approach (10). Sufficiently far from the flare this effect is nearly uniform. Observe the non-linear dependence between extracted mass and said overestimation: increasing the well input by a factor of 1.5 resulted in a fourfold prediction error, whilst a doubled input incurred a tenfold error. With the maximal permissible pressure commonly set slightly below atmospheric, e.g. 1250Pa, overestimation of up to 625Pa is significant. Whilst design guidelines usually suggest node vacuum of about 5kPa as a reasonable safety margin, in reality the operators strive to maintain the limit pressure as close to atmospheric as possible, forbearing to exert unnecessary suction liable to draw unsustainable amounts of landfill gas, beget air infiltration and decimation of the microorganisms responsible for biodegradation. The limit pressure is not set in stone and depends on the operator's experience with the fluctuations observed in each
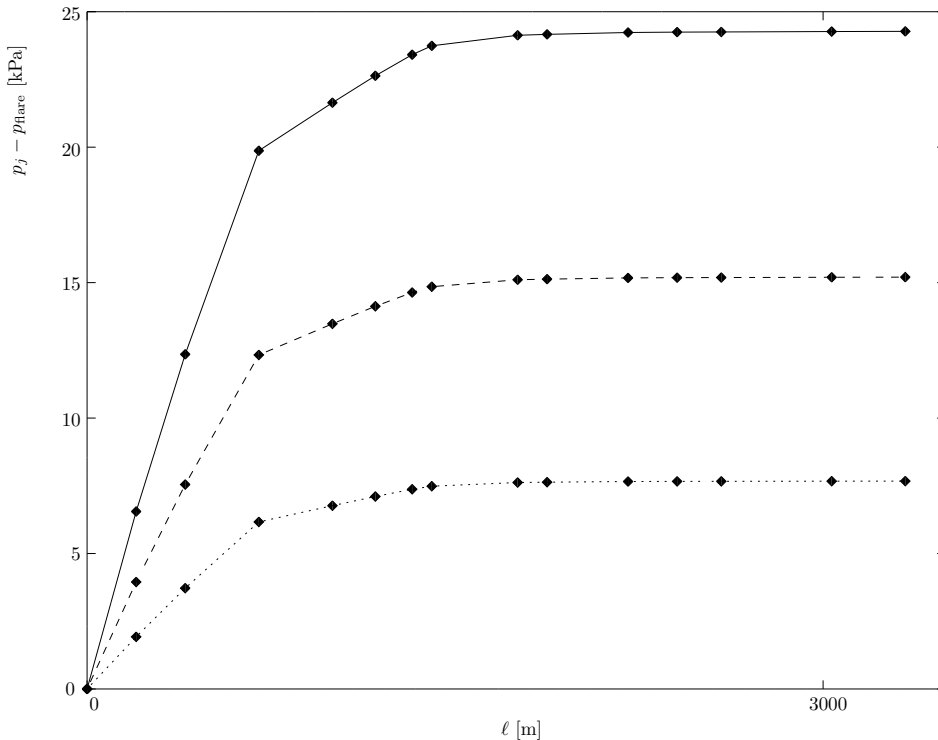
Figure 4: Pressure relatively to the flare along the path formed by pipes 1 to 14 (figure 3) for a total collection of $0.2\,\mathrm{kg/s}$ (dotted), $0.3\,\mathrm{kg/s}$ (dashed) and $0.4\,\mathrm{kg/s}$ (solid). Parameters used: flare node $i = 1$, $p_{\mathrm{flare}} = 25\mathrm{kPa}$ below atmospheric, $T = 13^{o}\mathrm{C}$, molar fractions $\left\{ x_{\mathrm{CH_4}}, x_{\mathrm{O_2}}, x_{\mathrm{CO_2}}, x_{\mathrm{N_2}} \right\} = \left\{ 0.5, 0.01, 0.45, 0.04 \right\}$.

locality. The extraction vacuum at the flare is usually significantly higher, although that too might vary widely during the lifetime of a landfill, when for instance the cover thickness or permeability is changed to accommodate addition of waste, landscaping etc. The ultimate goal in controlling these parameters is to retain a productive colony of anaerobic microorganisms.

From the vantage point of fluid mechanics the aforementioned findings are in congruence with the essence of error introduced due to misapplication of (10), since the computation is performed upstream from the flare (consult figure 2). Whilst the result accords well with intuition, it falls far from triteness, as owing to the presence of cycles in the network not every edge has the head change computed up the stream. In fact the flow direction along each edge was determined automatically every iteration of the solution. In a different application, where the reference pressure point is situated upstream, or where several reference points exist, a result of a different nature might ensue.

From the aspect of landfill design and operation the usage of (10) will result in more conservative predictions than necessary. The main complexity of the numerical solution of (24) and networks of this ilk stems from the non-linearity of pressure continuity equations (24b) as well as the relative number of fundamental cycles and edges in the
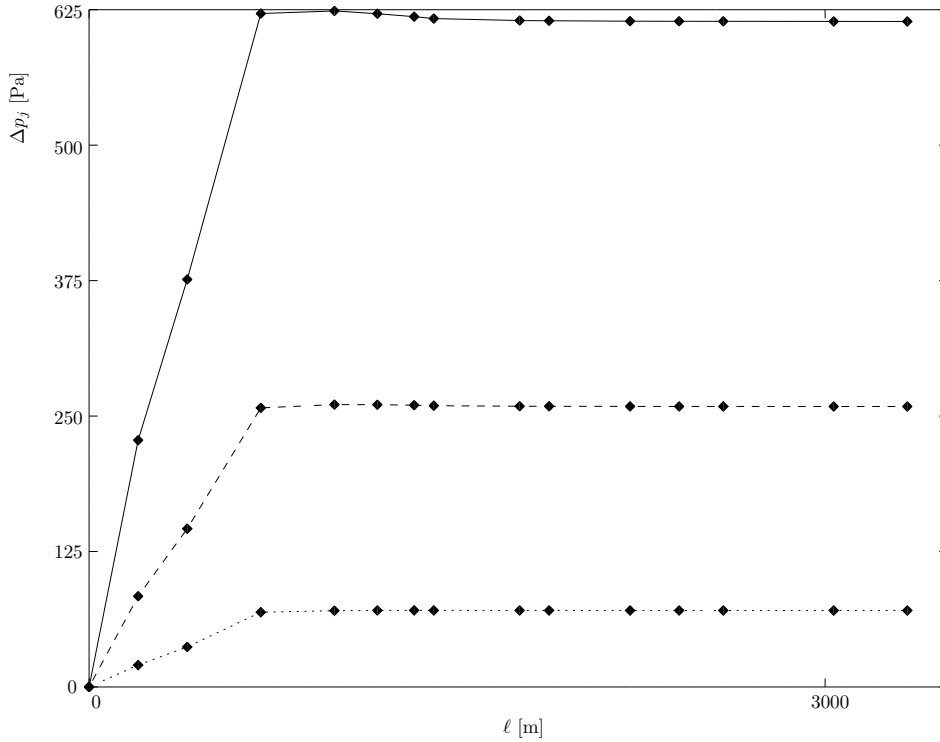
Figure 5: Variation between the classic and proposed methods: overestimation of pressure by the classic Darcy-Weisbach model (equation (10), constant density) versus the continuous model (equation (23), correctly varying density) along the path formed by pipes 1 to 14 (figure 3) for a total collection $0.2\,\mathrm{kg/s}$ (dotted), $0.3\,\mathrm{kg/s}$ (dashed) and $0.4\,\mathrm{kg/s}$ (solid). Parameters used: flare node $i = 1$, $p_{\mathrm{flare}} = 25\mathrm{kPa}$ below atmospheric, $T = 13^{o}\mathrm{C}$, molar fractions $\left\{x_{\mathrm{CH_4}}, x_{\mathrm{O_2}}, x_{\mathrm{CO_2}}, x_{\mathrm{N_2}}\right\} = \left\{0.5, 0.01, 0.45, 0.04\right\}$. For reference, maximal permissible pressure is commonly set slightly below atmospheric, e.g. 1250Pa.

graph. In practice the replacement of (10) with the more complicated at first glance (23) encumbers the solution but little (observe that a numerical solution of (2), for instance, and (23) demands the selfsame computational tools), however brings forth the advantage of a more accurate estimate. In particular, in order to direct the collected gas to beneficial uses the landfill operator is under an obligation to supply designated amounts of gas. As the maintained flow regime must comply with the maximal permissible pressure limit, overestimated pressure values will result in superfluous restrictions on the amount of gas extracted, undermining optimal collection and efficiency of the entire facility.

Figure 6 depicts the variation of Mach number along the same path as in figures 4 and 5. In this example, as in the vast majority of cases, the flare node would give the upper bound for the entire network. As expected, the Mach number is very low and nearly uniform in the part of the network containing interconnected cycles, climbing to the upper bound only in the acyclic part leading to the flare.
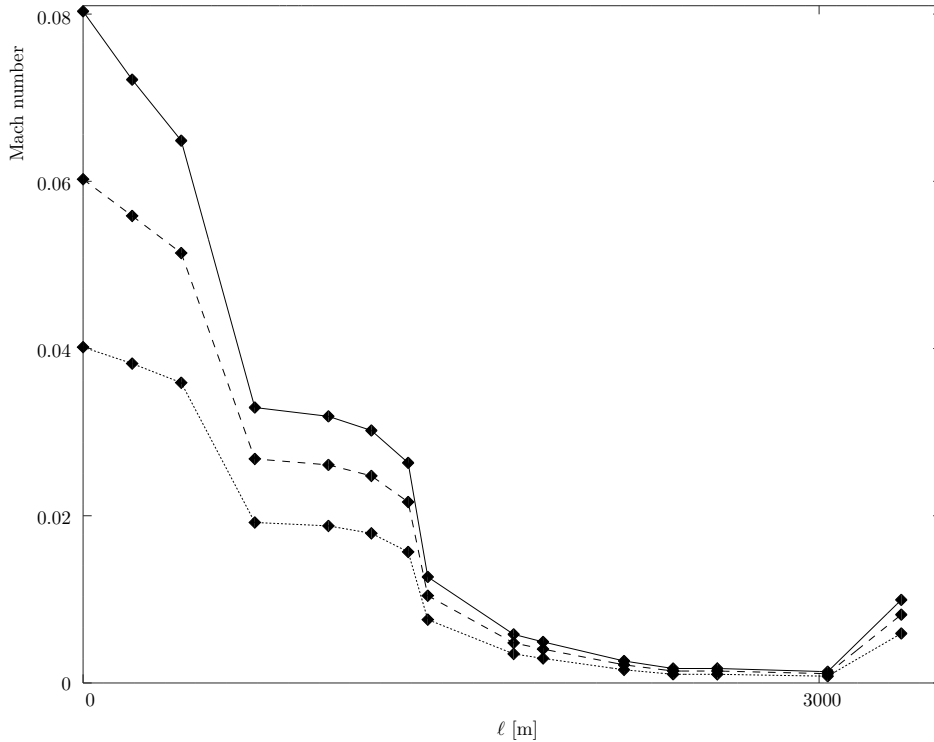
Figure 6: Mach number along the path formed by pipes 1 to 14 (figure 3) for a total collection of $0.2\,\mathrm{kg/s}$ (dotted), $0.3\,\mathrm{kg/s}$ (dashed) and $0.4\,\mathrm{kg/s}$ (solid). Parameters used: flare node $i = 1$, $p_{\mathrm{flare}} = 25\mathrm{kPa}$ below atmospheric, $T = 13^{o}\mathrm{C}$, molar fractions $\left\{x_{\mathrm{CH_4}}, x_{\mathrm{O_2}}, x_{\mathrm{CO_2}}, x_{\mathrm{N_2}}\right\} = \left\{0.5, 0.01, 0.45, 0.04\right\}$.

3.4. Kinked pipes

In landfill construction the occurrence of a pipe being kinked in the course of imbedment within the substratum covering the refuse cavity is a contingency to contend with. Mathematically a kinked pipe translates into elimination of the corresponding graph edge. Depending on the location, this modification might precipitate momentous changes for the entire network, possibly begetting flow reversal and flow rates manyfold larger than initially expected. One must bear in mind that when heavy construction machinery riding over the substratum damages a pipe laid within, the likelihood of repairs being effectuated is minuscule, rendering the resultant arrangement a permanent working regime. A gratuitously restrictive prediction of pressure distribution in the reconfigured system due to misapplication of (10) instead of (23) will necessitate an operator induced flow reduction at the wells in adherence to the limit of maximal permissible pressure in the system. Even small changes in the total flow rate annually amount to significant quantities of gas that might make a difference between the feasibility of a profitable alternative fuel or methane content enhancement facility, and the environmentally detrimental destruction by flaring with no useful work.

4. Discussion

A problem of flow in landfill gas collection network was considered. The fluid is an ideal gas and the regime isothermal, thereby the flow is weakly compressible. The traditional Darcy-Weisbach formula (1) developed for incompressible flows, whilst providing a close estimate in a certain part of the parameter space, is conceptually erroneous, the use thereof in conjunction with the state equation of ideal gas (3) setting forth an inherent incongruence. On the other hand, related formulae in wide use in the gas pipeline industry take into account compressibility effects insofar as to have recourse to multifarious empiric loss coefficients. The suggested method generalises the classic Darcy-Weisbach formula to include a continuous variation of density, yielding an ordinary differential equation. The approach is based on integral mass and momentum conservation equations, thence the result is obtained without undue premise on the constancy of the Darcy friction coefficient.

In the case of ideal gas the differential equation is soluble in closed form comprising non-linear algebraic equations (23), separate for the instances of horizontal and non-horizontal flow. The equation involves the traditional Darcy friction factor. Its use was justified by establishing an equivalence between a generic friction factor in the integral momentum conservation equation and empirically sound as well as easily accessible in practice Darcy friction factor. Any numerical algorithm suitable for solution of the Colebrook equation (2) to obtain f is equally capable of solving (23).

The network was characterised by an incidence matrix, permitting to implement the mass conservation equations at the nodes and pressure continuity equations in matrix form. The feasibility of (23) supplanting (10) was confirmed by solution of a flow in network geometry from a real landfill in the process of redesign and expansion. It was seen that little additional computational complexity arose due to the change of the head loss model, whereby it follows that the misapplication of the classic Darcy-Weisbach formula (10) for isothermal, compressible flow cannot be justified by the simplicity thereof. The solution stemming from the continuous approach herein properly accounts for compressibility, whose effect is non-negligible over significant pipe lengths. Furthermore, beyond the conceptually erroneous complexion of (10) in the context of isothermal gas flow, twofold pragmatic implications ensue. First, from the aspect of landfill operation the error in the estimation of pressure distribution in the network entails consistently more conservative working regime with reduced flow rates. Application of the conceptually correct (23) permits utilising the safety limits to their full potential. Second, extraction of all available gas and accurate estimate of the mass thereof allows for planning of environmentally more beneficial alternatives than flaring.

In summary, for a weakly compressible isothermal flow the Darcy-Weisbach formula is inapplicable. With the correct, although more involved, non-linear equation, within the context of pipe network analysis the additional computational complexity is minor. The suggested approach enables more accurate engineering decision making as regards efficient landfill operation and optimisation of gas collection.

pipe	diameter d_j [m]	length ℓ_j [m]
1	0.15	200
2	0.15	200
3	0.15	300
4	0.2	300
5	0.2	175
6	0.2	150
7	0.2	80
8	0.2	350
9	0.2	120
10	0.2	330
11	0.2	200
12	0.2	180
13	0.2	450
14	0.2	300
16	0.2	220
16	0.2	100
17	0.15	200
18	0.15	150
19	0.15	150
20	0.15	150
21	0.15	210
22	0.15	160
23	0.15	200
24	0.15	185
25	0.15	185
26	0.15	80
27	0.15	185
28	0.15	180

node	elevation h_i [m]	flow rate $\dot{m}^{(i)}$ [kg/s]
1	10	0
2	10	0.005
3	10	0.007
4	10	0.012
5	10	0.002
6	10	0.007
7	10	0.02
8	10	0.02
9	15	0.016
10	15	0.005
11	15	0.005
12	12	0.005
13	15	0.005
14	8	0.01
15	15	0.007
16	12	0.01
17	12	0.01
18	15	0.01
19	15	0.01
20	15	0.01
21	15	0.005
22	15	0.005
23	15	0.01

Table A1: Network details for the example in figure 3. Pipe data (left): internal pipe diameter d_j and effective pipe length ℓ_j, $j = \{1, \ldots, 28\}$. Node data (right): elevation h_i and input mass flow $\dot{m}^{(i)}$, $i = \{1, \ldots, 23\}$.

Appendix A. Example network details

Table A1 lists the pipe and node data used for the example network given in figure 3.

Acknowledgement

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