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Explicitly solvable eigenvalue problem and bifurcation delay in sub-diffusive Gierer–Meinhardt model

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A spike solution is constructed on the infinite line for a sub-diffusive version of the Gierer–Meinhardt reaction – diffusion model. A non-local eigenvalue problem governs the spike's stability and is explicitly solvable for a certain choice of the kinetic parameters. Its solution generalises former results for the Gierer–Meinhardt model with regular diffusion, and the normal and anomalous systems' properties are juxtaposed. It is shown that a Hopf bifurcation occurs in the sub-diffusive system for larger values of the time parameter τ_o as compared to the normal counterpart, rendering the anomalous system more stable. Asymptotic solutions are obtained near important values of the diffusion anomaly index γ and collectively shown to be valid over most of the applicable range $0 < \gamma < 1$. A bifurcation delay scenario is described for the sub-diffusive system, and the WKB exponent is computed analytically.

Key words: Sub-diffusion, reaction – diffusion, spike solution, non-local eigenvalue problem, bifurcation delay

1 Introduction

The current contribution was inspired by two recent developments in the field of anomalous reaction – diffusion systems. One, an activator–inhibitor system, known as the Gierer–Meinhardt model, had been endowed with a memory operator, resulting in an anomalously slow diffusion and modified asymptotic relation between the diffusivities of the two reagents [15]. In that study it was proved that the anomaly affects the drift and stability properties of patterns of localised solutions (spikes) on a finite domain. The salient conclusions of the analysis were three-fold: the anomalous pattern existed in a more realistic parameter regime than its normal counterpart, the symmetry of the leftward and rightward drift was broken due to the presence of memory, and the construction of a stability theory for such a system required the generalisation of some classic notions. The second breakthrough was achieved when an eigenvalue problem for the normal Gierer–Meinhardt model on an infinite line was solved explicitly for a particular set of parameters and the delayed onset of instability at a Hopf bifurcation point was analysed [18].

† Research for this contribution was conducted at Mount Allison University, New Brunswick, Canada.

Here it is the author's purpose to build upon the aforementioned results to find an explicit solution for the eigenvalue problem arising in the stability analysis of an anomalous Gierer–Meinhardt model on an infinite line and facilitate the comparison of bifurcation delay between the normal and anomalous systems. Because of the relative complexity of the topics of anomalous diffusion and the asymptotic analysis entailing the spike patterns, hereinafter each is reviewed separately.

The relevant type of anomalous diffusion and the corresponding mathematical operator are presented in Section 2. The Gierer–Meinhardt model is described in detail and the spike pattern is constructed in Section 3. The eigenvalue problem is derived and solved in Section 4, and the bifurcation delay is treated in Section 5. The conclusions are summarised in Section 6.

2 Anomalous diffusion and fractional derivatives

Diffusion anomalies have been discovered in sundry branches of the natural sciences circa three decades ago, and the experimental evidence has been accruing ever since. The solutions of some related mathematical equations had been first obtained even earlier [12], however as the moment of these was realised more widely, a plethora of models was constructed to describe the observed phenomena and motion properties particular to the specific system at hand. It is understood today that regular diffusion, characterised by a linear growth of the mean square displacement in time, i.e. $\langle r^2(t)\rangle \sim t$, is but a special limit of an infinity of processes that whilst having no universal description, can be classified into families [4]. Within the various scientific disciplines some of the models falling into any one of those generalised mathematical families can be obtained by narrowing the consideration to given system dynamics [9] and motion intricacies beyond the property of the mean square displacement [2].

One of the better studied anomalies ensues with the introduction of temporal memory into the random walk, resulting in a process with $\langle r^2(t)\rangle \sim t^\gamma$, $0 \leqslant \gamma < 1$, named sub-diffusion [11]. The dispersion of particles following such a process is signally limited in comparison to the regular diffusion, and if the diffusing particles are to participate in a concomitant process, for example, a chemical reaction, its rate is expected to diminish accordingly. The mathematical operator associated with such a process is a time fractional derivative of order γ.

The theory of fractional derivatives in general, alias fractional calculus, is related to the classic calculus much in the same way as sub-diffusion relates to Brownian motion: integer derivatives are but a special limit of a continuous family of operators, and as such manifest distinctive, although often taken for granted properties [17]. When a diffusion equation is endowed with a fractional derivative ∂_t^γ instead of the first time derivative ∂_t, most basic mathematical tools of calculus and theory of differential equations must be forfeited, a few immediate examples and their momentous consequences listed below.

- The kernel of the fractional derivative operator might be empty or spanned by other than the constant functions, thereby dispossessing a dynamical system of an equilibrium state: $\partial_t^\gamma\ 1 \not\equiv 0$, where 1 and 0 refer to unity and zero functions.
- The fractional derivative of an exponential function might not be an exponential,

wherefore the notion of operator spectrum and eigenmodes of a dynamical system is rendered meaningless: $\partial_t^{\gamma} e^t \neq e^t$.

- Successive application of several fractional derivatives is not equivalent to the application of a derivative of the summed order. In combination with the kernel properties, this makes the manipulation of fractional differential equations rather involute: $\partial_t^{\gamma_1} \partial_t^{\gamma_2} \neq \partial_t^{\gamma_1+\gamma_2}$.

Since in various contexts some properties are of more importance than others, there exist several types of fractional derivatives, chosen in accord with the demands of the required application. Regular differential operators have been replaced by fractional derivatives in many physical contexts, where it was desired to endow the system under discussion with memory. Since there is no universal description of any type of diffusion anomaly, the practice is to adopt a definition that upholds the most crucial properties of the system and then analyse the influence on less essential ones [7, 11]. Some models explore peculiar generalisations and the ensuing mathematical properties [1, 13]. All temporal fractional derivatives of order $0 < \gamma < 1$, as well as some other operators, give the desired global behaviour of the mean square displacement $\langle r^2(t) \rangle \sim t^{\gamma}$, and it is inevitable that other characteristics of the diffusive process will differ, in a marked difference to the regular diffusion. As long as the global behaviour constraint is satisfied, the choice of a suitable type of derivative is dictated by the context. In the current contribution, it is imperative that the fractional derivative of a constant function be the zero function. Therefore, the following definition is adopted [5, 15].

Definition 2.1 *Let $f(t)$ be a continuous function on $t > 0$. When its time fractional derivative of order γ exists, it is given by*

$$\frac{d^{\gamma}}{dt^{\gamma}} f(t) = -\frac{1}{\Gamma(-\gamma)} \int_0^t \frac{f(t) - f(t-\zeta)}{\zeta^{\gamma+1}} \, d\zeta, \qquad 0 < \gamma < 1, \tag{2.1}$$

wherein Γ denotes the Gamma function.

Remark 2.1 *The operator in (2.1) does not possess proper limits at $\gamma \longrightarrow 1^-$ and $\gamma \longrightarrow 0^+$. There exist definitions of fractional derivatives that do possess at least one of these limits [17], i.e. $\lim_{\gamma \longrightarrow 1^-} \frac{d^{\gamma} f}{dt^{\gamma}} = f'(t)$ and $\lim_{\gamma \longrightarrow 0^+} \frac{d^{\gamma} f}{dt^{\gamma}} = f(t)$. However, when acting on a constant function, those fractional derivatives do not give the zero function, thus rendering them unacceptable for the purpose of analysis of a dynamical system near an equilibrium that is not the quiescent solution. From the vantage point of physics, since the existence of stationary states underpins the analysis of reaction – diffusion systems with regular diffusion, Definition 2.1 is particularly suitable for the sub-diffusive counterpart.*

3 Spike solutions in Gierer–Meinhardt model with anomaly

In nature a mere dispersion of a set of particles is rare. Most often a diffusion process is sustained by a source or reaction, mathematically modelled as a non-linear term. When the diffusion is anomalous, care must be exercised when introducing reaction terms, so as to keep the operator positive definite. The reaction–diffusion system to be analysed in the current contribution is a paradigmatic model with spike-type solutions, where the ratio of diffusivities of the two species is asymptotically small. On an infinite line the sub-diffusive Gierer–Meinhardt model reads

$$\partial_t^\gamma a = \epsilon^{2\gamma} a_{xx} - a + \frac{a^p}{h^q} \qquad -\infty < x < \infty, \quad t > 0, \tag{3.1a}$$

$$\tau_o\, \partial_t^\gamma h = h_{xx} - h + \epsilon^{-\gamma} \frac{a^m}{h^s}, \qquad -\infty < x < \infty, \quad t > 0, \tag{3.1b}$$

$$\lim_{|x|\longrightarrow\infty} a = \lim_{|x|\longrightarrow\infty} h = 0, \qquad a(x,0) = a_0(x), \qquad h(x,0) = h_0(x), \tag{3.1c}$$

where $a(x,t)$ and $h(x,t)$ are the activator and inhibitor concentrations, respectively. Here $0 < \epsilon \ll 1$, $\tau_o > 0$, the quadruple of reaction exponents (p,q,m,s) satisfies

$$p > 1, \quad q > 0, \quad m > 0, \quad s \geqslant 0, \quad \frac{p-1}{q} < \frac{m}{s+1}, \tag{3.2}$$

and the anomaly exponent γ ranges $0 < \gamma < 1$. The set of kinetic exponents (p,q,m,s) is fixed for a given system. Due to the chemical origin of the model, the exponents are most often integers, however from a purely mathematical aspect an exponent might be any positive number as long as the conditions in (3.2) hold. In the course of the derivation, further conditions will be imposed to obtain explicit solutions [16].

The time constant τ_o is to be treated as a bifurcation parameter. Broadly speaking, the system is stable whilst τ_o is sufficiently small. As τ_o increases, a Hopf bifurcation is expected, and at sufficiently large values of τ_o the system possesses positive real eigenvalues. With regular diffusion, this behaviour is well known [18, 19]. Following a stability theory for a similar sub-diffusive system on a finite domain [15], analogous behaviour is to be obtained here.

The activator diffusivity of $\mathcal{O}\left(\epsilon^{2\gamma}\right)$ is essential for the activator concentration $a(x,t)$ to be localised about a set of loci, thus creating a pattern of spikes. The inhibitor diffusivity is on the order of unity, and being the faster diffusing species, its concentration $h(x,t)$ is not localised in the same fashion. It is worth noting that since both species diffuse with the same anomaly index γ, their ratio of diffusivities (and hence of the mean square displacement) remains constant. With the fractional derivative on the left-hand side, the reaction is diffusion limited, i.e. the reaction process is slowed to the same extent as the diffusion by the presence of memory, as the reaction encounters take place in accord with the pace the reagents find each other in the medium. Mathematically, this equal impediment of reaction and diffusion processes entails a positive definite operator, as opposed to a sub-diffusive formulation of the type $\partial_t u = \partial_t^{1-\gamma} u_{xx} + f(u)$ for some species u and source $f(u)$, where the diffusion is hindered, but not the reaction [7]. Broadly speaking, the reaction and diffusion terms cannot decouple, when the fractional derivative operator is inverted to give a derivative of order $1-\gamma$ on the right-hand side.

Substitution of $\gamma = 1$ in (3.1) recovers the Gierer–Meinhardt model with regular diffusion [6]. A similar system on a finite domain manifested $\mathcal{O}(1)$ as well as $\mathcal{O}\left(\epsilon^2\right)$ eigenvalues, and the analysis allowed for the spike drift [8]. In a normal counterpart of (3.1), regardless of the domain chosen, the spike pattern exists when the ratio of diffusivities is $\mathcal{O}\left(\epsilon^2\right)$, an unrealistic scenario for two substances diffusing in the same medium. In asymptotic theory, it is typical to expect the small parameter ϵ not to exceed 0.1 for a reasonable approximation to ensue. With ϵ thus small the regular Gierer–Meinhardt model requires a diffusivity ratio of at least 1:100, an extremely rare occurrence in natural processes. In the anomalous model, the ratio required for pattern existence is asymptotically better, and if γ is in the lower part of its allowed range $0 < \gamma < 1$, the ratio is absolutely realistic. For instance, with $\gamma = 0.2$, the required ratio $\epsilon^{2\gamma}$ will equal approximately 1:2.5. Technically, the compound ϵ^{γ} appears in many expressions and ostensibly is the small parameter involved in the asymptotic theory. However, the construction of the spike solution immediately hereinafter reveals a slow time scale that makes it convenient to retain the said compound in its current form. It must be remembered that this is not a simple change of the asymptotic scale, but an aftermath of the essentially slow diffusion, whose mean square displacement grows sub-linearly according to $\langle r^2(t)\rangle \sim t^{\gamma}$, i.e. no constant diffusion coefficient, however small, can entail this kind of impediment in the particle dispersion. If focussing on the mean square displacement alone, it might be said that the diffusion coefficient is time dependent and decays algebraically in time with the power $0 < 1-\gamma < 1$: $\langle r^2(t)\rangle \sim t^{\gamma-1}t$. This is what gives the more realistic asymptotic scaling and renders the analysis of this system of special interest.

3.1 Construction of a spike solution

Construction of a pattern of n spikes centred at a set of arbitrary loci x_i, $\{i = 0, \ldots, n-1\}$, is performed by the method of matched asymptotic expansions and follows closely the derivation of a similar pattern on a finite domain [15]. The outer solution for the activator away from the loci x_i is $a(x,t) \equiv 0$. The spatial variable of the inner layer is $y_i \stackrel{\text{def}}{=} (x-x_i)/\epsilon^{\gamma}$ with the corresponding slow time scale $\sigma = \epsilon^{\alpha} t$. The generalisation of the spatial scale is immediate when the differential term on the right-hand side in (3.1a) is sought to be $\mathcal{O}(1)$ within the inner layer. In contrast, the deduction of the corresponding temporal scale is not as simple.

In order to determine the temporal scale one must treat the problem of a spike's drift occurring on the slow time scale. The inner asymptotic solutions are set as

$$A_i(y_i,\sigma) = a\big(x_i + \epsilon^{\gamma} y_i, \epsilon^{-\alpha}\sigma\big) \sim A_i^{(0)}(y_i,\sigma) + \epsilon^{\gamma} A_i^{(1)}(y_i,\sigma) + \cdots \tag{3.3a}$$

$$H_i(y_i,\sigma) = h\big(x_i + \epsilon^{\gamma} y_i, \epsilon^{-\alpha}\sigma\big) \sim H_i^{(0)}(y_i,\sigma) + \epsilon^{\gamma} H_i^{(1)}(y_i,\sigma) + \cdots . \tag{3.3b}$$

From Definition 2.1, the fractional derivative of a function of a scaled variable $f(t) \equiv F(\sigma)$ gives

$$\partial_t^{\gamma} f(t) = \epsilon^{\alpha\gamma} \partial_{\sigma}^{\gamma} F(\sigma), \qquad F(\sigma) \equiv f(\epsilon^{-\alpha}\sigma). \tag{3.4}$$

Then substitution of (3.3) into (3.1) yields to leading order

$$\partial_{y_i}^2 A_i^{(0)} - A_i^{(0)} + \frac{A_i^{(0)p}}{H_i^{(0)q}} = 0, \qquad \lim_{|y_i|\longrightarrow\infty} A_i^{(0)} = 0 \tag{3.5a}$$

$$\partial_{y_i}^2 H_i^{(0)} = 0, \qquad \lim_{y_i\longrightarrow\pm\infty} H_i^{(0)} = \lim_{x\longrightarrow x_i^{\pm}} h^{(0)}. \tag{3.5b}$$

The boundary conditions in (3.5) were set by matching to the outer solution. Solving

(3.5b), it is found that $H_i^{(0)}$ must be independent of y_i. The reason is two-fold: due to the spike's symmetry H_i is expected to be an even function, and a linear function in y_i would not match with the outer solution. Then, (3.5a) is equivalent to

$$A_i^{(0)} = H_i^{(0)\beta} u(y_i), \quad \beta = \frac{q}{p-1}, \tag{3.6}$$

where $u(y)$ is the homoclinic solution of the non-linear ordinary differential equation

$$u'' - u + u^p = 0, \quad u'(0) = 0, \quad u(0) > 0, \quad \lim_{|y| \longrightarrow \infty} u(y) = 0, \tag{3.7}$$

explicitly solvable as

$$u(y) = \left\{ \left(\frac{p+1}{2}\right) \operatorname{sech}^2 \left(\frac{(p-1)}{2}\, y\right) \right\}^{1/(p-1)}. \tag{3.8}$$

Hence, $A_i(y_i, \sigma)$ decays exponentially as $|y_i| \longrightarrow \infty$ and is localised in the vicinity of the spike locus x_i.

The error in the leading order approximation is $\mathcal{O}(\epsilon^\gamma)$ in magnitude. At the next order, the problem for $A_i^{(1)}$ must involve the drift of the spike centre x_i [8]. With regular diffusion, $A_i^{(0)}(y_i(\sigma), \sigma)$ is differentiated with respect to the first argument, followed by the application of the chain rule introducing $\dfrac{dx_i}{d\sigma}$. The derivative with respect to the second argument is of a smaller order of magnitude. However, the chain rule does not hold when the derivative is fractional and one must proceed with caution.

Take $y_i(t) \equiv \big(x - x_i(\sigma)\big)/\epsilon^\gamma$ and assume $x_i \in C^\infty$. Further assume $A_i(y_i(\sigma)) \in C^\infty$. As long as a spike's shape remains intact, i.e. the neighbouring spikes' centres are sufficiently far away with only the exponentially small tails overlapping, these assumptions on the smoothness of solutions hold. By Definition 2.1,

$$\partial_\sigma^\gamma A_i(y_i(\sigma)) = -\frac{1}{\Gamma(-\gamma)} \int_0^\sigma \left\{ A_i\left(\frac{x - x_i(\sigma)}{\epsilon^\gamma}\right) - A_i\left(\frac{x - x_i(\sigma - \zeta)}{\epsilon^\gamma}\right) \right\} \frac{d\zeta}{\zeta^{\gamma+1}}. \tag{3.9}$$

A new variable ξ in terms of ζ is then defined by

$$\xi \equiv \big(x_i(\sigma - \zeta) - x_i(\sigma)\big)/\epsilon^\gamma. \tag{3.10}$$

To solve for ζ in terms of ξ when $\epsilon \ll 1$, x_i is expanded ($x_i \in C^\infty$) as

$$x_i(\sigma - \zeta) = x_i(\sigma) - \frac{dx_i}{d\sigma}\zeta + \frac{1}{2}\frac{d^2x_i}{d\sigma^2}\zeta^2 - + \ldots. \tag{3.11}$$

Hence, (3.10) becomes

$$\xi = \epsilon^{-\gamma}\left(-\frac{dx_i}{d\sigma}\zeta + \frac{1}{2}\frac{d^2x_i}{d\sigma^2}\zeta^2 - + \ldots\right). \tag{3.12}$$

Away from fixed points, where $\frac{dx_i}{d\sigma} = 0$, the series is reverted to give

$$\zeta = -\left(\frac{dx_i}{d\sigma}\right)^{-1}\left(\epsilon^{\gamma}\xi - \frac{1}{2}\frac{d^2x_i}{d\sigma^2}\zeta^2 + - \dots\right), \qquad \frac{dx_i}{d\sigma} \neq 0. \tag{3.13}$$

Note that this expression is exact – the series was not truncated, and the inversion holds everywhere except at the fixed points. A recursive substitution into higher powers of ζ yields

$$\zeta \sim \epsilon^{\gamma}\left(-\frac{dx_i}{d\sigma}\right)^{-1}\xi + \mathcal{O}\left(\epsilon^{2\gamma}\right). \tag{3.14}$$

Therefore, for $\epsilon \ll 1$ (3.9) becomes

$$\partial_{\sigma}^{\gamma}A_i\left(y_i(\sigma)\right) \sim -\frac{\epsilon^{-\gamma^2}}{\Gamma(-\gamma)}\left(-\frac{dx_i}{d\sigma}\right)^{-1}\int_0^{-\infty\cdot\mathrm{sgn}\left(\frac{dx_i}{d\sigma}\right)}\Big(A_i(y_i) - A_i(y_i-\xi)\Big)\left(-\frac{dx_i}{d\sigma}\frac{1}{\xi}\right)^{\gamma+1}d\xi. \tag{3.15}$$

Changing variables to have the upper integration bound positive,

$$\partial_{\sigma}^{\gamma}A_i\left(y_i(\sigma)\right) \sim -\epsilon^{-\gamma^2}\,\mathrm{sgn}\left(\frac{dx_i}{d\sigma}\right)\left|\frac{dx_i}{d\sigma}\right|^{\gamma}\mathfrak{D}_{y_i}^{\gamma}A_i(y_i), \tag{3.16a}$$

$$\mathfrak{D}_{y_i}^{\gamma}A_i(y_i) \equiv \mathrm{sgn}\left(\frac{dx_i}{d\sigma}\right)\frac{1}{\Gamma(-\gamma)}\int_0^{\infty}\left\{A(y_i) - A_i\left(y_i + \mathrm{sgn}\left(\frac{dx_i}{d\sigma}\right)\xi\right)\right\}\frac{d\xi}{\xi^{\gamma+1}}. \tag{3.16b}$$

Result (3.16) is the fractional equivalent of a chain rule in the case of an infinitely differentiable function A_i and in the limit $\epsilon \longrightarrow 0$.

Combining (3.16) and (3.6) to extract the equations for $A_i^{(1)}$ and $H_i^{(1)}$ from (3.1), it is found that the time scale must be set as $\alpha = \gamma + 1$ to have a proper order balance in the asymptotic solution. The corresponding time scale $\sigma = \epsilon^{\gamma+1}t$ evinces the impossibility of a simple replacement of the compound ϵ^{γ} used in many expressions by an equivalent small parameter. Whilst at first glance this underlying time scale is not obvious in the formulation (3.1), it is nonetheless significant both for the correctness of the current derivation and for congruence with the theory on drifting spikes in finite domains. Quite peculiarly, at the limit of regular diffusion ($\gamma = 1$), the diffusivity ratio $\epsilon^{2\gamma}$ and the time scale ratio $\epsilon^{\gamma+1}$ coincidentally equal, thereby creating an incorrect impression that with anomaly all asymptotic compounds will remain powers of one basic parameter ϵ^{γ}. To resume, the correction equations then are

$$H_i^{(0)-\beta}\mathscr{L}A_i^{(1)} = \frac{q}{H_i^{(0)}}u^p H_i^{(1)} - \mathrm{sgn}\left(\frac{dx_i}{d\sigma}\right)\left|\frac{dx_i}{d\sigma}\right|^{\gamma}\mathfrak{D}_{y_i}^{\gamma}u, \tag{3.17a}$$

$$\frac{\partial^2}{\partial y_i^2}H_i^{(1)} = -H_i^{(0)\beta m-s}u^m, \tag{3.17b}$$

where $\mathscr{L}$ is the linearised homoclinic operator $\frac{d^2}{dy^2} - 1 + pu^{p-1}$. The time derivative term

$\tau_o \partial_t^\gamma H_i^{(0)}$ in (3.1b) was neglected to obtain (3.17b). This is consistent when $\tau_o \partial_t^\gamma H_i^{(0)} \sim o(\epsilon^{-\gamma})$. Since $\partial_t^\gamma H_i^{(0)} = \epsilon^{\alpha\gamma} \partial_\sigma^\gamma H_i^{(0)}$ and $\alpha = \gamma + 1$, this condition holds when τ_o satisfies $\tau_o \sim o(\epsilon^{-\gamma(2+\gamma)})$.

By the Fredholm alternative, where $\mathscr{L}$ is self-adjoint, the solvability condition is

$$\int_{-\infty}^{\infty} \frac{du}{dy_i} \left\{ \frac{q}{H_i^{(0)}} u^p H_i^{(1)} - \operatorname{sgn} \frac{dx_i}{d\sigma} \left| \frac{dx_i}{d\sigma} \right|^\gamma \mathfrak{D}_{y_i}^\gamma u \right\} dy_i = 0. \tag{3.18}$$

The first term in (3.18) is integrated by parts twice and simplified with the aid of (3.7) along with the facts that u and $\partial_{y_i}^2 H_i^{(1)}$ are even functions. Then, the solvability condition becomes

$$\frac{q}{2(p+1)H_i^{(0)}} \int_{-\infty}^{\infty} u^{p+1} dy_i \left(\lim_{y_i \longrightarrow \infty} \frac{dH_i}{dy_i}^{(1)} + \lim_{y_i \longrightarrow -\infty} \frac{dH_i}{dy_i}^{(1)} \right)$$
$$= - \left| \frac{dx_i}{d\sigma} \right|^\gamma \operatorname{sgn} \left(\frac{dx_i}{d\sigma} \right) \int_{-\infty}^{\infty} \frac{du}{dy_i} \mathfrak{D}_{y_i}^\gamma u \, dy_i. \tag{3.19}$$

In the outer region away from the spike $a(x,t)$ is exponentially small and $h(x,t)$ is expanded as

$$h \sim h^{(0)}(x,t) + \mathcal{O}\left(\epsilon^\gamma\right).$$

To derive a differential equation for the outer problem for $h^{(0)}$, it is necessary to express the non-linear term in (3.1b) as a weighted δ-function due to the localised behaviour of a as

$$\epsilon^{-\gamma} \frac{a^m}{h^s} \sim \sum_{i=0}^{n-1} b_i \, \delta(x - x_i), \tag{3.20}$$

where the weight b_i is

$$b_i = \epsilon^{-\gamma} \int_{x_i^-}^{x_i^+} \frac{a^m}{h^s} \, dx = \int_{-\infty}^{\infty} \frac{a^m}{h^s} \, dy_i \sim H_i^{(0)\beta m - s} \int_{-\infty}^{\infty} u^m \, dy.$$

Then from (3.1b), the outer equation for $h^{(0)}$ is

$$h_{xx}^{(0)} - h^{(0)} = b_m \sum_{i=0}^{n-1} H_i^{(0)\beta m - s} \delta(x - x_i), \qquad b_m = \int_{-\infty}^{\infty} u^m \, dy. \tag{3.21}$$

missing minus before b_m

Equation (3.21) is solved as

$$h^{(0)}(x,t) = -b_m \sum_{i=0}^{n-1} H_i^{(0)\beta m - s} G(x; x_i), \tag{3.22}$$

where the Green's function

$$G(x; x_i) = -\frac{1}{2} e^{-|x - x_i|} \tag{3.23}$$

is the solution of

$$G_{xx} - G = \delta(x - x_i), \qquad \lim_{|x| \longrightarrow \infty} G(x; x_i) = 0. \tag{3.24}$$

Then with

$$\lim_{y_i\longrightarrow\infty}\partial_{y_i}H_i^{(1)}+\lim_{y_i\longrightarrow-\infty}\partial_{y_i}H_i^{(1)}=\lim_{x\longrightarrow x_i^+}h_x^{(0)}+\lim_{x\longrightarrow x_i^-}h_x^{(0)}$$

$$=b_m\left\{2\sum_{\substack{j=0\\j\neq i}}^{n-1}H_j^{(0)\beta m-s}G_x(x_i;x_j)+H_i^{(0)\beta m-s}\Big(G_x(x_i^-;x_i)+G_x(x_i^+;x_i)\Big)\right\},\quad(3.25)$$

(3.19) becomes

$$\operatorname{sgn}\left(\frac{dx_i}{d\sigma}\right)\left|\frac{dx_i}{d\sigma}\right|^{\gamma}=-\frac{qb_mf(p;\gamma)}{(p+1)H_i^{(0)}}\left\{\frac{1}{2}H_i^{(0)\beta m-s}\Big(G_x(x_i^-;x_i)+G_x(x_i^+;x_i)\Big)+\sum_{\substack{j=0\\j\neq i}}^{n-1}H_j^{(0)\beta m-s}G_x(x_i;x_j)\right\},\quad(3.26a)$$

$$H_i^{(0)}(\sigma)=b_m\sum_{j=0}^{n-1}H_j^{(0)\beta m-s}G(x_i;x_j),\quad(3.26b)$$

$$f(p;\gamma)\equiv\left(\int_{-\infty}^{\infty}u^{p+1}\,dy\right)\Big/\left(\int_{-\infty}^{\infty}u'(y)\mathfrak{D}_y^{\gamma}u\,dy\right).\quad(3.26c)$$

Hereinafter, the analysis will focus on a single spike centred at an arbitrary point x_o. In this particular case, it is readily inferred upon matching the inner and outer solutions that

$$h^{(0)}(x_o,t)=H^{(0)}=\left(\frac{2}{b_m}\right)^{1/(\beta m-s-1)}.\quad(3.27)$$

Hence, the globally valid asymptotic solutions for the activator and inhibitor concentrations are

$$a_{\text{eq}}\sim H^{(0)\beta}\,u\left(\frac{x-x_o}{\epsilon^{\gamma}}\right)\quad(3.28a)$$

$$h_{\text{eq}}\sim -b_mH^{(0)\beta m-s}G(x;x_o).\quad(3.28b)$$

The spike is at equilibrium due to symmetry: if the obtained Green's function (3.23) is substituted into the drift equation of a generic pattern (3.26), it is found explicitly that the spike locus is stationary, i.e.

$$\frac{dx_o}{d\sigma}=0,\quad(3.29)$$

and for this degenerate case the only effect of anomaly is the altered width of the spike. However, for patterns of more spikes the new time scale comes into play along with the broken symmetry of a leftward/rightward drift, as is seen from the fractional operator (3.16*b*).

The immediate question regarding the stability of this spike is treated in Section 4.

4 Eigenvalue problem

A stability theory for the anomalous spike constructed in Section 3 requires certain painstakingness. Since the fractional derivative 2.1 of an exponential function is not an exponential, no exponentially evolving disturbances can be expected when (3.1) is linearised. Therefore, if a comparison between (3.1) and its normal limit with $\gamma = 1$ is to be effected, the focus must be narrowed to a class of disturbances that are exponential to leading order:

$$a \sim a_{\text{eq}} + e^{\lambda t}\,\tilde{a}(x), \quad \tilde{a}(x) \sim \tilde{a}^{(0)} + \epsilon^{\gamma}\tilde{a}^{(1)} + \cdots, \quad |\tilde{a}| \ll 1, \tag{4.1a}$$

$$h \sim h_{\text{eq}} + e^{\lambda t}\,\tilde{h}(x), \quad \tilde{h}(x) \sim \tilde{h}^{(0)} + \epsilon^{\gamma}\tilde{h}^{(1)} + \cdots, \quad |\tilde{h}| \ll 1, \tag{4.1b}$$

$$\lambda(t) \sim \lambda^{(0)} + \epsilon^{\gamma}\lambda^{(1)}(t) + \cdots, \quad \lambda^{(0)} = \text{const.}$$

Here, a_{eq}, h_{eq} are the solutions in (3.28). The exponent $\lambda^{(0)}$ constitutes an anomalous eigenvalue for the purpose of comparison to the classic eigenvalue of the normal system. In the special limit of regular diffusion, the disturbance is purely exponential, whence $\lambda^{(1)}$ and all subsequent corrections must vanish. Unfortunately, the analytical difficulty of fractional calculus prevents the determination of $\lambda^{(1)}$ explicitly, and the analysis is confined to the determination of $\lambda^{(0)}$. Moreover, at this stage it is not obvious that the class of perturbations considered is the most unstable, however the quest for disturbances of disparate behaviour is beyond the scope of the current contribution. The purpose of this section is to facilitate a comparison between the stability of (3.1) and its normal counterpart giving rise solely to exponential perturbations. Hereinafter, an eigenvalue problem for $\tilde{a}^{(0)}$ is to be derived, where $\tilde{a}^{(0)}$ is regarded as an anomalous eigenfunction. Henceforth $\{\lambda^{(0)}, \tilde{a}^{(0)}\}$ is referred to as an eigenvalue–eigenfunction pair for convenience, yet one must bear in mind that only in the limit $\gamma = 1$ do they in fact correspond to these classic notions.

Upon substitution of the disturbance form (4.1) into (3.1), linearisation and subsequent cancellation of the exponential terms, the following expression ensues:

$$e^{-\lambda^{(0)}t}\frac{d^{\gamma}}{dt^{\gamma}}e^{\lambda^{(0)}t} = -\frac{1}{\Gamma(-\gamma)}\int_0^t \frac{1 - e^{-\lambda^{(0)}\zeta}}{\zeta^{\gamma+1}}\,d\zeta. \tag{4.2}$$

The integral on the right hand side is convergent as $t \longrightarrow \infty$ if and only if $\Re\lambda^{(0)} \geqslant 0$, and furthermore in the limit $\epsilon \longrightarrow 0$ the following asymptotics is obtained [15]:

$$e^{-\lambda^{(0)}t}\frac{d^{\gamma}}{dt^{\gamma}}e^{\lambda^{(0)}t} \sim \lambda^{(0)\gamma} + \mathcal{O}\left(\epsilon^{\gamma(\gamma+1)}\right). \tag{4.3}$$

Estimate (4.3) possesses a proper limit at $\gamma \longrightarrow 1^-$ and recovers the normal relation upon the substitution $\gamma = 1$, whereat it is exact and the error term ought to be omitted. The restriction $\Re\lambda^{(0)} \geqslant 0$ prevents the tracing of eigenvalues in the left half plane, however since for the regular diffusion system, studied in depth in the last thirty years [8, 19, 20], trajectories have rarely been constructed for the motion of eigenvalues prior to the onset of instability by a Hopf bifurcation [3], the impossibility to follow the eigenvalues in the left half plane is not detrimental for the purport of comparison at hand. Moreover, as will become evident hereunder, the part of the complex plane, where an anomalous eigenvalue may exist is the sector $|\arg z| < \gamma\pi$, whilst the instability zone is $|\arg z| < \frac{1}{2}\gamma\pi$. These two sectors are the generalised notions of the complex plane and the right half plane respectively. The Hopf bifurcation will occur on the border $|\arg z| = \frac{1}{2}\gamma\pi$, replacing the imaginary axis. The reason behind these changes will become apparent once it is shown that the anomalous eigenvalue problem can be mapped onto the normal one by $\lambda^{(0)^\gamma} \longmapsto \lambda$ (equation (4.3) being a harbinger of such an inference), and hence if a normal eigenvalue lies on the principal branch, i.e. $-\pi < \arg\lambda \leqslant \pi$, perforce the anomalous concominant $\lambda^{(0)^\gamma}$ must satisfy $-\gamma\pi < \arg\lambda^{(0)} \leqslant \gamma\pi$. The instability zone and the Hopf bifurcation locus are mapped identically.

Linearising (3.1) about the equilibrium solution (3.28) and using (4.3) yields

$$\lambda^{(0)^\gamma}\tilde{a}^{(0)} = \left(\epsilon^{2\gamma}\frac{d^2}{dx^2} - 1 + p\frac{a_{\text{eq}}^{p-1}}{h_{\text{eq}}^{q}}\right)\tilde{a}^{(0)} - q\frac{a_{\text{eq}}^{p}}{h_{\text{eq}}^{q+1}}\tilde{h}^{(0)}, \tag{4.4a}$$

$$\tau_o\lambda^{(0)^\gamma}\tilde{h}^{(0)} = \left(\frac{d^2}{dx^2} - 1 - \epsilon^{-\gamma}s\frac{a_{\text{eq}}^{m}}{h_{\text{eq}}^{s+1}}\right)\tilde{h}^{(0)} + \epsilon^{-\gamma}m\frac{a_{\text{eq}}^{m-1}}{h_{\text{eq}}^{s}}\tilde{a}^{(0)}, \tag{4.4b}$$

$$\lim_{|x|\longrightarrow\infty}\tilde{a}^{(0)} = \lim_{|x|\longrightarrow\infty}\tilde{h}^{(0)} = 0. \tag{4.4c}$$

From the activator equation (4.4a) an eigenfunction is expected with a localisation similar to that of the equilibrium solution (3.28a)

$$\tilde{a} \sim \tilde{a}^{(0)}\left(\frac{x-x_o}{\epsilon^\gamma}\right) + \mathcal{O}(\epsilon^\gamma). \tag{4.5}$$

In the inhibitor equation (4.4b) the non-linear terms can be represented as weighted δ-functions similarly to (3.20), resulting in the following problem:

$$\frac{d^2}{dx^2}\tilde{h}^{(0)} - \left(1 + \tau_o\lambda^{(0)^\gamma} + sb_m H^{(0)^{\beta m-s-1}}\delta(x-x_o)\right)\tilde{h}^{(0)} = -w_o\delta(x-x_o), \tag{4.6}$$

$$w_o = mH^{(0)^{\beta(m-1)-s}}\int_{-\infty}^{\infty} u^{m-1}\tilde{a}^{(0)}\,dy.$$

Problem (4.6) with boundary conditions (4.4c) is equivalent to the following problem defined on the real line with continuity and jump conditions at the spike centre

$$\frac{d^2}{dx^2}\tilde{h}^{(0)} - \left(1 + \tau_o\lambda^{(0)^\gamma}\right)\tilde{h}^{(0)} = 0, \qquad \lim_{|x|\longrightarrow\infty}\tilde{h}^{(0)} = 0, \tag{4.7a}$$

$$\tilde{h}^{(0)}\left(x_o^+\right) = \tilde{h}^{(0)}\left(x_o^-\right), \qquad \left(\frac{d}{dx}\tilde{h}^{(0)}\Big|_{x_o^+} - \frac{d}{dx}\tilde{h}^{(0)}\Big|_{x_o^-}\right) = sb_m H^{(0)^{\beta m-s-1}}\tilde{h}^{(0)}(x_o) - w_o, \tag{4.7b}$$

readily solved as

$$\tilde{h}^{(0)} = \frac{w_o\,e^{-\mu|x-x_o|}}{2(\mu+s)}, \qquad \mu = \sqrt{1+\tau_o\lambda^{(0)^\gamma}}. \tag{4.8}$$

Substituting (4.5) into (4.4a) gives

$$\frac{d^2}{dy^2}\tilde{a}^{(0)} - \left(1 + \lambda^{(0)^\gamma} - pu^{p-1}\right)\tilde{a}^{(0)} = qH^{(0)^{\beta-1}}u^p\tilde{h}^{(0)}(x_o), \qquad \lim_{|y|\longrightarrow\infty}\tilde{a}^{(0)} = 0, \tag{4.9}$$

thereby with the use of (4.8) cast into

$$\mathscr{L}\,\tilde{a}^{(0)} - u^p\,\frac{\chi}{b_m}\int_{-\infty}^{\infty} u^{m-1}\,\tilde{a}^{(0)}dy = \lambda^{(0)^\gamma}\,\tilde{a}^{(0)}, \qquad \lim_{|y|\longrightarrow\infty}\tilde{a}^{(0)} = 0, \tag{4.10}$$

$$\mathscr{L} = \frac{d^2}{dy^2} - 1 + pu^{p-1}, \quad \chi = \frac{mq}{\sqrt{1+\tau_o\lambda^{(0)^\gamma}}+s}.$$

In this form the non-local eigenvalue problem (4.10) is identical to the one derived for a single spike on a finite domain [15], only the multiplier χ reflects the difference in geometry. Problem (4.10) also conforms to the counterpart with regular diffusion [18] upon the mapping $\lambda^{(0)^\gamma} \longmapsto \lambda$, thereat explicitly solvable if $p = 2m - 3$, the solution being given by

$$\lambda^{(0)^\gamma} = m\Big(m - 2 - \frac{\chi}{2}\Big), \tag{4.11}$$

or equivalently

$$\sqrt{1+\tau_o\lambda^{(0)^\gamma}} = \mathscr{G}\left(\lambda^{(0)}\right), \qquad \mathscr{G}\left(\lambda^{(0)}\right) \stackrel{\text{def}}{=} \frac{m^2 q}{2\Big(\mathfrak{m} - \lambda^{(0)^\gamma}\Big)} - s, \quad \mathfrak{m} = m(m-2), \tag{4.12}$$

so that the root $\lambda^{(0)}$ can be obtained by seeking the intersection point of $\mathscr{G}$ and the square root function on the left-hand side. With regular diffusion, the properties of these two functions allowed for certain conclusions regarding the existence and uniqueness of this intersection [18]. Furthermore, by imposing a particular form on $\lambda^{(0)}$ the Hopf bifurcation was found. Hereinafter performing an analysis similar in concept, yet more complicated due to the more generic nature of the anomalous diffusion, the anomalous Hopf bifurcation point is sought.

With the presence of anomaly, the salient feature of the two functions in (4.12) deviating significantly from the normal behaviour is the infinite derivative at $\lambda^{(0)} \longrightarrow 0^+$. This is a manifestation of the improper limit at $\gamma \longrightarrow 1^-$, as with the substitution of $\gamma = 1$ the slopes of both functions are finite at $\lambda^{(0)} \longrightarrow 0^+$. $\mathscr{G}$ still has an asymptote, though with anomaly it is situated at $\lambda_{\mathrm{a}}^{(0)} = \mathfrak{m}^{1/\gamma}$. Since $\mathscr{G}' > 0\ \forall\ 0 < \lambda^{(0)} < \lambda_{\mathrm{a}}^{(0)}$, there must be an inflexion point within this interval, which is absent in the regular counterpart. The function $\sqrt{1+\tau_o\lambda^{(0)^\gamma}}$ is concave just like with regular diffusion.

With regular diffusion when the time parameter τ_o is tuned, a pair of complex conjugate eigenvalues moves on the complex plane as follows [18]: for $0 < \tau_o \ll 1$ there are no eigenvalues in the right half plane, at $\tau_o = \tau_H$ a pair crosses the imaginary axis, for $\tau_H < \tau_o < \tau_m$ there is a complex conjugate pair, merging onto the real axis at $\tau_o = \tau_m$, and two real positive eigenvalues for $\tau_o > \tau_m$. Since the anomalous problem (4.10) might be obtained from the normal counterpart by the mapping $\lambda \longmapsto \lambda^{(0)^\gamma}$, seeking an anomalous eigenvalue with $\arg\lambda^{(0)} = \varphi$ is equivalent to solving the normal problem for an eigenvalue with $\arg\lambda = \gamma\varphi$. Hence, a trajectory of the motion across the imaginary axis and towards the real axis must still ineluctably be traced, only the thresholds and the progress velocity will differ. To find the new Hopf bifurcation threshold set $\lambda^{(0)} = \imath\lambda_H$ in (4.12) to obtain

$$\mathscr{G}\Big(\imath\lambda_H\Big) = \sqrt{1+\tau_H\left(\imath\lambda_H\right)^\gamma}, \qquad \lambda_H \in \mathbb{R}, \tag{4.13}$$

which is equivalent to having $\lambda = \exp(\frac{1}{2}\imath\gamma\pi)$.

It must be noted that the mapping $\lambda \longmapsto \lambda^{(0)\gamma}$ suffices to infer that the anomalous system is more stable than the concomitant with simple diffusion: when the anomalous eigenvalue first crosses the imaginary axis, its normal counterpart has already traversed a part of the right-half plane and is located at $\arg\lambda = \frac{1}{2}\gamma\pi$. Hence, the system with regular diffusion is in an unstable regime for all τ_o exceeding the normal bifurcation threshold, whereas with anomaly the instability only just sets in.

The remnant of this section is dedicated to finding the anomalous Hopf bifurcation point by exact as well as asymptotic solutions of (4.13). The solutions extend the cases covered hitherto for regular diffusion [18] to a wider range of kinetic exponents as well as to the full range of the anomaly exponent γ.

4.1 Hopf point for s=0

Solving (4.13) analytically is somewhat tedious, so the simpler case of $s = 0$ is treated first. Using the identity $|z^\alpha| = |z|^\alpha$, equate the moduli on the two sides of (4.13) to yield

$$\left(1 + 2\tau_H \lambda_H^\gamma \cos\tilde{\gamma} + \tau_H^2 \lambda_H^{2\gamma}\right)^{1/4} = \frac{m^2 q}{2\left(\mathrm{m}^2 - 2\mathrm{m}\lambda_H^\gamma \cos\tilde{\gamma} + \lambda_H^{2\gamma}\right)^{1/2}}, \quad \tilde{\gamma} = \frac{\pi\gamma}{2}. \tag{4.14}$$

Using the identity $\arg z^\alpha = \alpha \arg z$ equate the arguments on the two sides of (4.13) to yield

$$\frac{1}{2}\,\mathrm{arctg}\,\frac{\tau_H \lambda_H^\gamma \sin\tilde{\gamma}}{1 + \tau_H \lambda_H^\gamma \cos\tilde{\gamma}} = \mathrm{arctg}\,\frac{\lambda_H^\gamma \sin\tilde{\gamma}}{\mathrm{m} - \lambda_H^\gamma \cos\tilde{\gamma}}. \tag{4.15}$$

With

$$\mathrm{tg}(2\alpha) = \frac{2\,\mathrm{tg}\,\alpha}{1 - \mathrm{tg}^2\,\alpha}$$

and elementary algebra (4.15) gives the explicit expression for $\tau_H(\lambda_H)$

$$\frac{1}{\tau_H} = \frac{\left(\mathrm{m} - 2\lambda_H^\gamma \cos\tilde{\gamma}\right)^2 - \lambda_H^{2\gamma}}{2\left(\mathrm{m} - \lambda_H^\gamma \cos\tilde{\gamma}\right)}. \tag{4.16}$$

Squaring (4.14) and utilising (4.16) to simplify yields

$$\left(\mathrm{m}^2 - 2\mathrm{m}\lambda_H^\gamma \cos\tilde{\gamma} + \lambda_H^{2\gamma}\right)^2 = \frac{m^4 q^2}{2\tau_H}\left(\mathrm{m} - \lambda_H^\gamma \cos\tilde{\gamma}\right), \tag{4.17}$$

and upon elimination of τ_H results in a quartic for λ_H^γ

$$\lambda_H^{4\gamma} - 4\mathrm{m}\cos\tilde{\gamma}\,\lambda_H^{3\gamma} + \left({}_2M_{\frac{1}{4}} + {}_4M_{-1}\cos^2\tilde{\gamma}\right)\lambda_H^{2\gamma} - \mathrm{m}\,{}_4M_{-1}\cos\tilde{\gamma}\,\lambda_H^\gamma + \mathrm{m}^2\,{}_1M_{-\frac{1}{4}} = 0,$$

$${}_aM_b(m,q) \stackrel{\mathrm{def}}{=} a\,\mathrm{m}^2 + b\,m^4 q^2. \tag{4.18}$$

With $\gamma = 1$ equation (4.18) reduces to a quadratic in λ_H^2 since then $\cos\tilde{\gamma} = 0$. The system with exponents $(p,q,m,s) = (3,2,3,0)$ has been analysed in depth [18] and is a particular case of (4.18). The full quartic is, of course, solvable analytically, however it will be virtually impossible to adjudge which root out of the four is to be chosen. Therefore, it is advisable to obtain asymptotic solutions near important values of γ such as $\gamma = 1$ (regular diffusion), $\gamma = 0$ (the most extreme anomalous case) and $\gamma = \frac{1}{2}$ (representative fractional value). From (4.18) it is evident that an asymptotic solution for λ_H^γ in the vicinity of any value of γ must be a power series of the form

$$\lambda_H^\gamma = \lambda_o\Big(1 + \varepsilon\,\frac{\lambda_1}{\lambda_o} + \varepsilon^2\,\frac{\lambda_2}{\lambda_o} \;+\; \cdots\Big), \qquad |\varepsilon| \ll 1 \tag{4.19}$$

with ε unrelated to the small diffusivity parameter ϵ. However, it so happens that $\lambda_1 < 0$ for a substantial set of γ within $0 < \gamma < 1$, and thus according to (4.19) to first order λ_H has a root at $\varepsilon \approx -\lambda_o/\lambda_1$. In reality λ_H has no such root, but for (4.19) to be accurate ε must be very small or else many terms in the series must be computed. It was found that a power series form

$$\lambda_H = \lambda_o\Big(1 + \varepsilon\,\frac{\lambda_1}{\lambda_o} + \varepsilon^2\,\frac{\lambda_2}{\lambda_o} \;+\; \cdots\Big), \qquad |\varepsilon| \ll 1 \tag{4.20}$$

approximates the true solution more accurately for a wider range of ε. In the following sections for each of $\gamma = 0,\ \frac{1}{2},\ 1$, an asymptotic solution is accordingly obtained to leading order followed by two corrections. For particular choices of the exponents (p, q, m, s), the asymptotic solutions are compared to the full numerical solution of (4.18).

4.1.1 *Asymptotic solution near $\gamma = 1$*

Taking $\gamma = 1 - \varepsilon$, $0 < \varepsilon \ll 1$, expanding

$$\cos\tilde{\gamma} \sim \frac{\pi}{2}\,\varepsilon - \frac{\pi^3}{48}\,\varepsilon^3 + \mathcal{O}\big(\varepsilon^5\big) \tag{4.21a}$$

$$\lambda_H^\gamma \sim \lambda_o + \Big(\lambda_1 - \lambda_o \ln\lambda_o\Big)\varepsilon + \Big(\lambda_2 - \lambda_1(1 + \ln\lambda_o) + \frac{1}{2}\lambda_o \ln^2\lambda_o\Big)\varepsilon^2 + \mathcal{O}\big(\varepsilon^3\big), \tag{4.21b}$$

substituting into (4.18) and collecting similar powers of ε gives

$$\lambda_o = \sqrt{\;-\,{}_1M_{\frac{1}{8}} + \frac{m^2 q}{2}\sqrt{2\,{}_1M_{\frac{1}{32}}}\;}\;, \tag{4.22a}$$

$$\lambda_1 = \lambda_o \ln\lambda_o + \pi m\;\frac{2\lambda_o^2 \;+\; {}_2M_{-\frac{1}{2}}}{4\lambda_o^2 \;+\; {}_4M_{\frac{1}{2}}}, \tag{4.22b}$$

$$\lambda_2 = \lambda_1\big(1 + \ln\lambda_o\big) - \frac{\lambda_o}{2}\;\ln^2\lambda_o \;+\; \frac{\pi m g\left(6\lambda_o + \dfrac{{}_2M_{-\frac{1}{2}}}{\lambda_o}\right) - g^2\left(6\lambda_o + \dfrac{{}_2M_{\frac{1}{4}}}{\lambda_o}\right) - \pi^2\;{}_1M_{-\frac{1}{4}}\lambda_o}{4\lambda_o^2 \;+\; {}_4M_{\frac{1}{2}}}, \tag{4.22c}$$

$$g = \lambda_1 - \lambda_o \ln\lambda_o.$$

The leading order solution (4.22*a*) corresponds to regular diffusion, and for the particular case $(p, q, m, s) = (3, 2, 3, 0)$ has been formerly obtained [18].

4.1.2 *Solution for* $\gamma = \frac{1}{2}$

Taking $\gamma = \frac{1}{2}$ in (4.18) and rearranging into a more compact form gives

$$\left(\lambda_H - \sqrt{2}\,\mathfrak{m}\sqrt{\lambda_H} + \mathfrak{m}^2\right)^2 = \frac{m^4 q^2}{4}\left(\lambda_H - 2\sqrt{2}\,\mathfrak{m}\sqrt{\lambda_H} + \mathfrak{m}^2\right). \tag{4.23}$$

Equation (4.23) is a quartic in $\sqrt{\lambda_H}$. It is possible to solve (4.23) by following some of the steps known for such equations. First add into the parentheses on the left hand side an arbitrary $\mathfrak{y}$, balance the right hand side accordingly and choose $\mathfrak{y}$ so that the right hand side becomes a complete square. This yields a cubic for $\mathfrak{y}$

$$\mathfrak{y}^3 + {}_1M_{\frac{1}{8}}\,\mathfrak{y}^2 - \frac{1}{32}m^8 q^4 \mathfrak{m}^2 = 0. \tag{4.24}$$

The sequence of transformations $\mathfrak{y} = \tilde{\mathfrak{y}} - \frac{1}{3}\left({}_1M_{\frac{1}{8}}\right)$ and $\tilde{\mathfrak{y}} = w + \left({}_1M_{\frac{1}{8}}\right)^2 \Big/ (9w)$ yields an immediately solvable quadratic for w^3

$$w^6 + \left(\frac{2}{27}\left({}_1M_{\frac{1}{8}}\right)^3 - \frac{1}{32}m^8 q^4 \mathfrak{m}^2\right)w^3 + \frac{1}{3^6}\left({}_1M_{\frac{1}{8}}\right)^6 = 0. \tag{4.25}$$

Hence upon solving (4.25) for w and reversing the sequence of transformations to obtain $\mathfrak{y}$, a quadratic for $\sqrt{\lambda_H}$ ensues by taking the root of (4.23), where the right hand side has been cast in a complete square form by the addition of $\mathfrak{y}$

$$\lambda_H - \sqrt{2}\,\mathfrak{m}\sqrt{\lambda_H} + \mathfrak{m}^2 + \mathfrak{y} = \pm\sqrt{\frac{1}{4}m^4 q^2 + 2\mathfrak{y}}\left(\sqrt{\lambda_H} - \alpha\right), \quad \alpha = \frac{\sqrt{2}\,\mathfrak{m}}{2}\,\frac{\frac{1}{2}m^4 q^2 + 2\mathfrak{y}}{\frac{1}{4}m^4 q^2 + 2\mathfrak{y}}. \tag{4.26}$$

The solution λ_H is immediate and is omitted here due to its cumbersome form. It ought to be noted that equation (4.25) yields six roots, which in pairs give three roots for $\mathfrak{y}$. It is proved below that there exists exactly one solution $\mathfrak{y} > 0$. The discriminant of (4.25) is negative regardless of m and q, thus perforce w is complex. Nonetheless, there exists a single real positive $\mathfrak{y}$. With this value $\mathfrak{y} > 0$ one of the roots of (4.26) conforms to the desired Hopf bifurcation eigenvalue.

To prove the existence of a unique $\mathfrak{y} > 0$ define

$$g_1(\mathfrak{y}) = \mathfrak{y}^2\left(\mathfrak{y} + {}_1M_{\frac{1}{8}}\right) \tag{4.27}$$

and rearrange (4.24) to read

$$g_1(\mathfrak{y}) = \frac{1}{32}m^8 q^4 \mathfrak{m}^2. \tag{4.28}$$

The function g_1 has a double root and a minimum at $\mathfrak{y} = 0$. Its remaining root is at $\mathfrak{y} = -\,{}_1M_{\frac{1}{8}} < 0$. Hence, g_1 is ascending for all $\mathfrak{y} > 0$ and will have a unique intersection prescribed by (4.28). This suffices for the purport of procuring one positive value $\mathfrak{y}$, thence entailing real coefficients in the quadratic (4.26).

It is further possible to determine when additional real values of $\mathfrak{y}$ exist. This is useful in numerical implementation by way of caution. The function g_1 has a maximum at $\mathfrak{y} = -\frac{2}{3}({}_1M_{\frac{1}{8}})$ and the maximal value is $\max g_1 = \frac{4}{27}({}_1M_{\frac{1}{8}})^3$. Hence if $\max g_1 < \frac{1}{32}m^8q^4\mathfrak{m}^2$, the real root to (4.28) is unique. If however

$$\max g_1 \geqslant \frac{1}{32}m^8q^4\mathfrak{m}^2, \tag{4.29}$$

there will be two negative (equal when the equality is taken) roots. Equation (4.29) can be rearranged into

$$\left(\mathfrak{y}_1 + \frac{1}{8}\right)^3 \geqslant \frac{27}{128}\mathfrak{y}_1, \qquad \mathfrak{y}_1 = \left(\frac{m-2}{mq}\right)^2. \tag{4.30}$$

Define

$$g_2(\mathfrak{y}_1) = \left(\mathfrak{y}_1 + \frac{1}{8}\right)^3 - \frac{27}{128}\mathfrak{y}_1 = \left(\mathfrak{y}_1 - \frac{1}{4}\right)\left(\mathfrak{y}_1^2 + \frac{5}{8}\mathfrak{y}_1 - \frac{1}{128}\right). \tag{4.31}$$

By the initial constraint on the reaction exponents (3.2) with $s = 0$ and $p = 2m - 3$, it follows that $\mathfrak{y}_1 < \frac{1}{2}$. Thus by $g_2 \leqslant 0$ all roots of (4.28) will be real (with only one positive) if

$$\frac{1}{4} \leqslant \frac{m-2}{mq} < \frac{1}{2} \quad \text{or} \quad 0 < \frac{m-2}{mq} \leqslant \frac{\sqrt{-5+\sqrt{27}}}{4}, \tag{4.32}$$

and there will be exactly one real root (positive) if

$$\frac{\sqrt{-5+\sqrt{27}}}{4} < \frac{m-2}{mq} < \frac{1}{4}. \tag{4.33}$$

4.1.3 *Asymptotic solution near* $\gamma = \frac{1}{2}$

With the Hopf bifurcation eigenvalue given by (4.26) it is possible to obtain an asymptotic solution near $\gamma = \frac{1}{2}$. Here too the power series form (4.20) is preferable. Taking $\gamma = \frac{1}{2} + \varepsilon$, $|\varepsilon| \ll 1$, expanding

$$\cos\tilde{\gamma} \sim \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{2}\,\varepsilon - \frac{\pi^2}{8}\,\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right) \tag{4.34a}$$

$$\lambda_H^\gamma \sim \sqrt{\lambda_o}\left\{1 + \left(\ln\lambda_o + \frac{\lambda_1}{\lambda_o}\right)\varepsilon + \frac{1}{2}\left(\left(\ln\lambda_o + \frac{\lambda_1}{\lambda_o}\right)^2 + \frac{2\lambda_1}{\lambda_o} + \frac{\lambda_2}{\lambda_o} - \frac{\lambda_1^2}{2\lambda_o^2}\right)\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right\}, \tag{4.34b}$$

substituting into (4.18) and collecting similar powers of ε gives

$$\lambda_1 = -2\lambda_o\left(\ln\lambda_o + \pi\,\frac{\sqrt{2}\,\mathfrak{m}\lambda_o - \sqrt{\lambda_o}\;{}_2M_{-\frac{1}{2}} + \sqrt{2}\,\mathfrak{m}\;{}_1M_{-\frac{1}{4}}}{4\lambda_o^{3/2} - 6\sqrt{2}\,\mathfrak{m}\lambda_o + \sqrt{\lambda_o}\;{}_8M_{-\frac{1}{2}} - \sqrt{2}\,\mathfrak{m}\;{}_2M_{-\frac{1}{2}}}\right), \tag{4.35a}$$

$$\lambda_2 = \frac{\lambda_1^2}{2\lambda_o} - 2\lambda_1 - \lambda_o g^2 - \lambda_o \times$$

$$\frac{6\lambda_o^{3/2}g^2 - 2\sqrt{2}\,\mathfrak{m}\lambda_o\Big(3g^2 - \tfrac{3\pi}{2}g - \tfrac{\pi^2}{8}\Big) + \sqrt{\lambda_o}\,g(\,{}_4M_{-\frac{1}{4}}\,g - \pi\,{}_4M_{-1}) + \pi\sqrt{2}\,\mathfrak{m}\,{}_1M_{-\frac{1}{4}}\Big(g + \tfrac{1}{4}\Big)}{2\lambda_o^{3/2} - 3\sqrt{2}\,\mathfrak{m}\lambda_o + \sqrt{\lambda_o}\,{}_4M_{-\frac{1}{4}} - \sqrt{2}\,\mathfrak{m}\,{}_1M_{-\frac{1}{4}}}, \tag{4.35b}$$

$$g = \ln\lambda_o + \frac{\lambda_1}{2\lambda_o},$$

where in all of the above λ_o is the solution from (4.26).

4.1.4 *Asymptotic solution near* $\gamma = 0$

Sufficiently far from $\gamma = 1$ the inaccuracy problem due to a retained power form (4.19) becomes irrelevant. In particular near $\gamma = 0$ the power series form (4.20) is inapplicable. Taking $0 < \varepsilon = \gamma \ll 1$, from the expansion

$$\cos\tilde{\gamma} \sim 1 - \frac{\pi^2}{8}\gamma^2 + \frac{\pi^4}{384}\gamma^4 + \mathcal{O}\big(\gamma^6\big) \tag{4.36a}$$

it becomes obvious that the appropriate asymptotic form must be

$$\lambda_H^\gamma \sim \lambda_o + \gamma^2\lambda_2 + \gamma^4\lambda_4 + \mathcal{O}\big(\gamma^6\big). \tag{4.36b}$$

As before, substituting into (4.18) and collecting similar powers of γ gives at order $\mathcal{O}\big(\gamma^0\big)$ a quartic for λ_o

$$\Big(\lambda_o - \mathfrak{m}\Big)^4 = \frac{m^4q^2}{4}\Big(3\lambda_o - \mathfrak{m}\Big)\Big(\lambda_o - \mathfrak{m}\Big). \tag{4.37}$$

The point $(\mathfrak{m}, 0)$ in the complex plane conforms to the eigenvalue of the local operator $\mathscr{L}$ in (4.10) with an eigenfunction of a constant sign [20]. In the non-local problem all eigenvalues are located to the left of that point. Hence, $\lambda_o \neq \mathfrak{m}$ and (4.37) becomes

$$\Big(\lambda_o - \mathfrak{m}\Big)^3 = \frac{m^4q^2}{4}\Big(3\lambda_o - \mathfrak{m}\Big). \tag{4.38a}$$

At higher orders

$$\lambda_2 = -\frac{\pi^2}{8}\,\lambda_o\,\frac{\mathfrak{m}\Big(\lambda_o - \mathfrak{m}\Big)^2 + \frac{1}{2}\,m^4q^2\Big(\lambda_o - \frac{1}{2}\mathfrak{m}\Big)}{\Big(\lambda_o - \mathfrak{m}\Big)^3 - \frac{3}{8}\,m^4q^2\Big(\lambda_o - \frac{2}{3}\mathfrak{m}\Big)}, \tag{4.38b}$$

$$\lambda_4 = \frac{1}{4}\Bigg\{ - 6\lambda_2^2\bigg(\Big(\lambda_o - \mathfrak{m}\Big)^2 - \frac{1}{8}m^4q^2\bigg) + \pi^2\lambda_2\bigg(- \frac{3}{2}\,\mathfrak{m}\lambda_o^2 + \lambda_o\,{}_2M_{-\frac{1}{2}} - \mathfrak{m}\,{}_{\frac{1}{2}}M_{-\frac{1}{8}}\bigg)$$

$$+\frac{\pi^4}{96}\,\lambda_o\bigg(\mathfrak{m}\lambda_o^2 - 2\lambda_o\,{}_4M_{-1} + \mathfrak{m}\,{}_1M_{-\frac{1}{4}}\bigg)\Bigg\}\Bigg/\Bigg\{(\lambda_o - \mathfrak{m})^3 - \frac{3}{8}\,m^4q^2\Big(\lambda_o - \frac{2}{3}\mathfrak{m}\Big)\Bigg\}, \tag{4.38c}$$

where in all of the above λ_o is the solution of (4.38*a*) given by

$$\lambda_o = \mathfrak{m} + w + \frac{m^4q^2}{4w}, \qquad w^3 = \frac{m^4q^2}{4}\Big(\mathfrak{m} \pm \sqrt{\,{}_1M_{-\frac{1}{4}}}\,\Big). \tag{4.39}$$

Similarly to the situation with $\gamma = \frac{1}{2}$ the variable w is complex, nonetheless λ_o is real. In particular if $0 < \lambda_o < 1$, the solution at $\gamma = 0$ is the infimum

$$\lim_{\gamma \longrightarrow 0^+} \lambda_H = \lim_{\gamma \longrightarrow 0^+} \left(\lambda_o + \gamma^2 \lambda_2 + \gamma^4 \lambda_4 + \mathcal{O}(\gamma^6) \right)^{1/\gamma} = 0. \tag{4.40}$$

On the other hand, if $\lambda_o > 1$, the solution becomes unbounded as $\gamma \longrightarrow 0^+$, since the limit in (4.40) does not exist. This happens for sufficiently high reaction exponents m (and consequently p since $p = 2m - 3$), a phenomenon formerly observed in a problem similar to (4.10), where the distinction was made according to the magnitude of the eigenvalue of the local operator [15]. There the local eigenvalue was denoted v_{loc} and did not equal m since the constraint $p = 2m - 3$ was not imposed to solve the non-local eigenvalue problem explicitly. Nonetheless, $v_{\text{loc}} > 1$ at the limit $\gamma \longrightarrow 0^+$ resulted, technically, in an infinite Hopf bifurcation frequency. These artefacts are ultimately related to the invalidity of the initial asymptotic expansions in (3.3) at the limit $\gamma \longrightarrow 0^+$. Therefore, approximation (4.36*b*) is in effect as along as γ is small, but not at the limit $\gamma \longrightarrow 0^+$ itself, whereby ϵ in (3.3) can tend to zero with the validity of the expansion intact.

4.1.5 *Combining asymptotic solutions for the range* $0 \leqslant \gamma \leqslant 1$

The numerical solution to (4.18) was compared to the asymptotic solutions in the vicinity of $\gamma = 0, \frac{1}{2}, 1$ in Figure 1. The asymptotic solution near $\gamma = 1$, equation (4.22), is virtually indistinguishable from the true solution over the interval $\gamma \in (0.8, 1)$. The asymptotic solution near $\gamma = 0$, equation (4.38), is accurate over the interval $\gamma \in (0, 0.5)$. The intermediate asymptotic solution near $\gamma = \frac{1}{2}$, equation (4.35), gives a good approximation over the interval $\gamma \in (0.3, 0.7)$. Hence, solutions (4.38) and (4.35) have a significant region of overlap, whereas to obtain the same for (4.22) and (4.35) more terms in the asymptotic expansions are required. The difference stems from the fact that (4.38) had the correct functional form retained, whilst (4.22) and (4.35) were cast into a power series form, the correct functional form giving less accurate estimates for a series truncated after only a few terms.

Figure 1 is typical for $m = 3$ and any q. As q is increased, two phenomena are observed. One, the interval of γ between $\gamma = 1$ and $\gamma = \frac{1}{2}$, where the asymptotics (4.22) and (4.35) fail to capture the true solution, grows. Two, the interval, where the asymptotics (4.38) approximates the true solution accurately, although developed for $0 < \gamma \ll 1$, shrinks slightly.

The corresponding Hopf bifurcation threshold τ_H is shown in Figure 2. The sub-diffusive spike is more stable than the normal counterpart as is readily seen from the dependence $\tau_H(\gamma)$: as the anomaly index γ diminishes, τ_H grows, i.e. the point of instability onset occurs for higher τ_H in the anomalous system. Note the monotonicity of τ_H in γ, which accords well with the underlying physical mechanism of this type of sub-diffusion, namely memory. The dispersion of the reagents depends on the entire history of the motion of each component, effectively slowing down all processes, reaction as well as diffusion. Higher values of the time constant τ_o essentially mean that the interaction is sped up, and

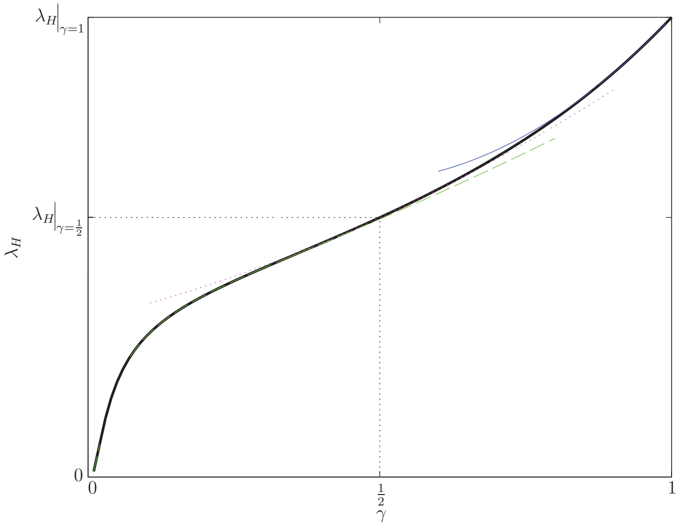

FIGURE 1. Hopf bifurcation point λ_H for $(p, q, m, s) = (3, 2, 3, 0)$: thick solid curve – full numerical solution of (4.18); thin solid curve – asymptotic solution near $\gamma = 1$, equation (4.22); dashed curve – asymptotic solution near $\gamma = 0$, equation (4.38); dotted curve – asymptotic solution near $\gamma = \frac{1}{2}$, equation (4.35).

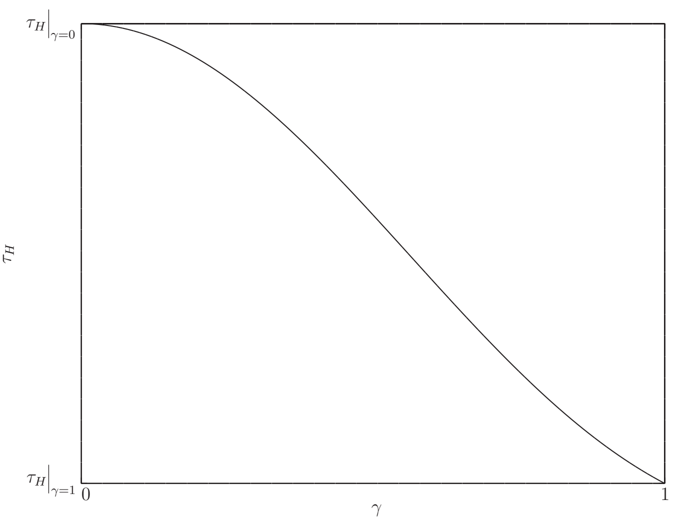

FIGURE 2. Hopf bifurcation threshold τ_H for $(p, q, m, s) = (3, 2, 3, 0)$ versus anomaly exponent γ.

this increase must compensate for the overall hindered diffusion and reaction in order for the bifurcation to occur.

When m is increased, the limit (4.40) ceases to exist since $\lambda_o > 1$ in (4.39). Quite expectedly, it is still possible to employ the asymptotic solutions (4.22) and (4.35) for $\gamma \geqslant \frac{1}{2}$. For $\gamma < \frac{1}{2}$, the bifurcation frequency λ_H quickly exceeds computable values. Rather intriguingly, it is possible to use the asymptotic solution (4.38), developed for $0 < \gamma \ll 1$, for $\gamma \geqslant \frac{1}{2}$. Apparently, the full functional form is captured so faithfully that the accuracy

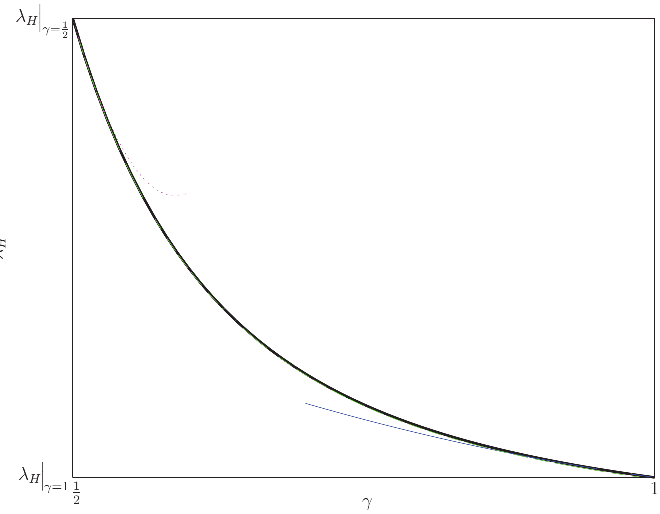

FIGURE 3. Hopf bifurcation point λ_H for $(p,q,m,s) = (9,2,6,0)$: thick solid curve – full numerical solution of (4.18); thin solid curve – asymptotic solution near $\gamma = 1$, equation (4.22); dashed curve – asymptotic solution near $\gamma = 0$, equation (4.38), indistinguishable from the true solution over the entire interval $\gamma \in (0.5, 1)$; dotted curve – asymptotic solution near $\gamma = \frac{1}{2}$, equation (4.35).

is surprising: if $m - q \geqslant 4$, this solution is indistinguishable from the true one over the entire interval $\gamma \in (0,1)$. Figure 3 depicts a typical example for $\gamma \in (\frac{1}{2}, 1)$. For $\gamma < \frac{1}{2}$, the function $\lambda_H(\gamma)$ blows up exponentially according to (4.36*b*).

4.2 Hopf point for $s > 0$

Re-writing equation (4.13) for a generic value of s as

$$\sqrt{1 + \tau_H \big(\imath\lambda_H\big)^{\gamma}} = \frac{\frac{1}{2}m^2 q}{\mathfrak{m} - \big(\imath\lambda_H\big)^{\gamma}} - s, \tag{4.41}$$

squaring, separating the real and imaginary parts and simplifying yields a system of two real equations in τ_H and λ_H:

$$2\tau_H(1+\cos\tilde{\gamma})\lambda_H^{3\gamma} + \lambda_H^{2\gamma}(1-s^2-2\mathfrak{m}\tau_H) - \left(\mathfrak{m}(1-s) + \frac{1}{2}m^2 q\right)\left(\mathfrak{m}(1+s) - \frac{1}{2}m^2 q\right) = 0, \tag{4.42a}$$

$$\big(4\cos^2\tilde{\gamma} - 1\big)\tau_H\lambda^{2\gamma} + 2\cos\tilde{\gamma}\,\lambda_H^{\gamma}\big(1 - s^2 - 2\mathfrak{m}\tau_H\big) + \mathfrak{m}^2\tau_H - 2\mathfrak{m}(1-s^2) - m^2 qs = 0. \tag{4.42b}$$

It is possible to isolate τ_H from (4.42*b*) to read

$$\frac{1}{\tau_H} = \frac{\Big(\mathfrak{m} - 2\lambda_H^{\gamma}\cos\tilde{\gamma}\Big)^2 - \lambda_H^{2\gamma}}{m^2 qs + 2(1-s^2)\Big(\mathfrak{m} - \lambda_H^{\gamma}\cos\tilde{\gamma}\Big)}, \tag{4.43}$$

which as expected upon the substitution of $s = 0$ recovers equation (4.16).

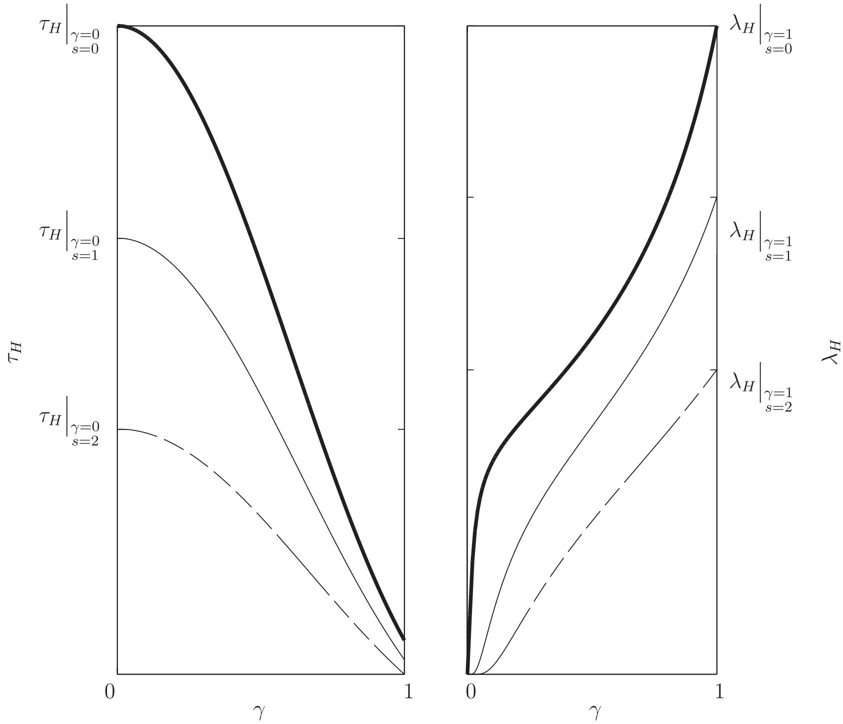

FIGURE 4. A typical influence of the kinetic parameter s on the Hopf bifurcation point λ_H (right) and threshold τ_H (left). Kinetic exponents used $(m, p, q) = (3, 3, 3)$ and $s = 0$ (thick solid curves), $s = 1$ (thin solid curves) and $s = 2$ (dashed curves).

If (4.43) is plugged into (4.42a), a quartic in λ_H^γ will ensue, generalising (4.18). It is then possible to repeat the analysis of Section 4.1 and generalise all results for $s > 0$, however the algebraic complexity of that is scarcely instructive. Therefore, to determine the influence of the kinetic exponent s on the Hopf bifurcation point λ_H and threshold τ_H system (4.42) was solved numerically. Figure 4 depicts the typical influence of s on the graphs $\lambda_H(\gamma)$ and $\tau_H(\gamma)$. One must bear in mind that for a set of kinetic exponents (p, q, m), the values of s are limited by inequality (3.2). It is seen that there is little qualitative distinction between the curves conforming to the various values of s.

5 Hopf bifurcation delay

The concept of passage through a bifurcation point as a means of exploration of the point's characteristics is not new [10, 14]. The effect of delay when crossing a Hopf bifurcation point for system (3.1) with regular diffusion was recently addressed [18]. In a classic setting the control parameter τ_o is to cross slowly the Hopf bifurcation threshold τ_H. Introduce a slow time scale $\tau = \varepsilon t$, $|\varepsilon| \ll 1$ (independent of the small diffusivity ϵ) to have $\tau_o = \tau_H + \tau$ and accordingly a disturbance of the type

$$a = a_{\text{eq}} + e^{\psi(\tau)/\epsilon}\varphi(x), \quad h = h_{\text{eq}} + e^{\psi(\tau)/\epsilon}\eta(x), \qquad |\varphi|, |\eta| \ll 1, \tag{5.1}$$

where $a_{\rm eq}$, $h_{\rm eq}$ are the equilibrium solutions in (3.28). Substituting (5.1) into (3.1) will require the fractional derivative of the exponential function $\exp\left(\psi(\tau)/\epsilon\right)$. It was proved formerly that the fractional derivative (2.1) of a simple exponential $\exp(\lambda t)$ exists only if the exponent λ has a positive real part [15]. An anologous constraint for a WKB type exponent in (5.1) is obtained below.

Applying the operator in (2.1) to $\exp\left(\psi(\tau)/\epsilon\right)$ and changing the integration variable from t to the slow scale τ yields

$$\frac{d^\gamma}{dt^\gamma}\, e^{\psi(\tau)/\epsilon} = -\frac{\operatorname{sgn}\epsilon|\epsilon|^\gamma e^{\psi(\tau)/\epsilon}}{\Gamma(-\gamma)} \int_0^\tau \frac{1-e^{\left(\psi(\tau-\xi)-\psi(\tau)\right)/\epsilon}}{|\xi|^{\gamma+1}}\, d\xi. \tag{5.2}$$

Assuming $\psi(\tau)$ possesses a Taylor expansion convergent for $\tau \in \mathbb{R}$, equation (5.2) becomes

$$\frac{d^\gamma}{dt^\gamma}\, e^{\psi(\tau)/\epsilon} = -\frac{\operatorname{sgn}\epsilon|\epsilon|^\gamma e^{\psi(\tau)/\epsilon}}{\Gamma(-\gamma)} \int_0^\tau \frac{1-e^{\psi'(\tau)\left(-\xi+\frac{1}{2}\xi^2\frac{\psi''(\tau)}{\psi'(\tau)}-+\cdots\right)/\epsilon}}{|\xi|^{\gamma+1}}\, d\xi, \tag{5.3}$$

where the series in the numerator is taken in its entirety and does not constitute an approximation for small ξ. Bear in mind that τ might be negative, however $\operatorname{sgn}\tau = \operatorname{sgn}\epsilon$. Therefore for the integral to converge with an infinite upper bound $\psi'(\tau) > 0$ must hold. This constraint is a generalisation of the former demand that the exponent of a simple exponential be positive [15].

Changing the integration variable again to $\zeta = \psi'(\tau)\xi/\epsilon$ gives

$$\frac{d^\gamma}{dt^\gamma}\, e^{\psi(\tau)/\epsilon} = -\frac{e^{\psi(\tau)/\epsilon}}{\Gamma(-\gamma)}\, \psi'^\gamma(\tau) \int_0^{\tau\psi'(\tau)/\epsilon} \frac{1-e^{-\zeta+\frac{1}{2}\sigma\frac{\psi''(\tau)}{\psi'^2(\tau)}\zeta^2-+\cdots}}{\zeta^{\gamma+1}}\, d\zeta. \tag{5.4}$$

Generally, a fractional derivative of an exponential function is not an exponential, however here the derivative at the limit $\epsilon \longrightarrow 0$ is required. Equation (5.4) is integrated by parts and it is observed that

$$\frac{1-e^{-\zeta+\frac{1}{2}\epsilon\frac{\psi''(\tau)}{\psi'^2(\tau)}\zeta^2-+\cdots}}{\zeta^\gamma}$$

vanishes at the limit $\zeta \longrightarrow 0$ by L'Hôpital's rule and at the upper bound $\zeta = \tau\psi'(\tau)/\epsilon$, since all terms in the infinite series are of order $\mathcal{O}\left(\epsilon^{-1}\right)$. Hence

$$\lim_{\epsilon\longrightarrow 0}\frac{d^\gamma}{dt^\gamma}\, e^{\psi(\tau)/\epsilon} = \lim_{\epsilon\longrightarrow 0}\frac{e^{\psi(\tau)/\epsilon}}{\Gamma(1-\gamma)}\, \psi'^\gamma(\tau) \int_0^{\tau\psi'(\tau)/\epsilon} \frac{e^{-\zeta+\frac{1}{2}\epsilon\frac{\psi''(\tau)}{\psi'^2(\tau)}\zeta^2-+\cdots}}{\zeta^\gamma}\left(1-\epsilon\,\frac{\psi''(\tau)}{\psi'^2(\tau)}+\mathcal{O}\left(\epsilon^2\right)\right)d\zeta$$

$$= \lim_{\epsilon\longrightarrow 0}\frac{e^{\psi(\tau)/\epsilon}}{\Gamma(1-\gamma)}\, \psi'^\gamma(\tau)\int_0^\infty e^{-\zeta}\zeta^{-\gamma}d\zeta = \psi'^\gamma(\tau)\, \lim_{\epsilon\longrightarrow 0} e^{\psi(\tau)/\epsilon}, \tag{5.5}$$

where the last equality was obtained with the definition of the Gamma function. The ultimate limit in (5.5) must exist by the postulated disturbance form (5.1).

Using (5.5) in (3.1) and linearising, it is found that $\psi'(\tau)$ satisfies the same non-local eigenvalue problem (4.10) as the generalised eigenvalue $\lambda^{(0)}$, solvable explicitly as before to give

$$\psi'^{\gamma}(\tau) = \mathfrak{m} - \frac{m^2 q}{2\left(s + \sqrt{1 + \tau_o(\tau)\psi'^{\gamma}(\tau)}\right)}, \tag{5.6}$$

generalising an analogous finding with regular diffusion [18].

5.1 Delay scenario

Hereinafter the derivation follows closely the method used for the analysis of delay in (3.1) with regular diffusion [18] to facilitate an apposite comparison between the two systems. The delay scenario is as follows: τ_o is set to an initial value $\tau_{\text{init}} < \tau_H$ and then slowly increased past the threshold τ_H according to the slow time scale τ. When $\tau_o = \tau_{\text{init}}$, $\tau = 0$ and hence (5.6) is supplemented by the initial condition $\psi(0) = 0$. Since $\psi'(\tau) < 0$ as long as $\tau_o < \tau_H$ (eigenvalue in the left half plane), $\psi(\tau)$ is negative. As soon as τ_o crosses τ_H, $\psi'(\tau)$ changes sign (Hopf bifurcation occurs and the eigenvalue begins its motion into the right half plane), and $\psi(\tau)$ starts growing. So conceptually integration of $\psi'(\tau)$ ought at some $\tau_* > \tau_H$ to reach a point, where $\Re\psi(\tau_*) = 0$. Using the correspondence between ψ' and λ, ψ' was integrated analytically and a transcendental equation was obtained to be solved for τ_*, the point beyond which the disturbance will for the first time have a positive real part.

With anomaly caution is required since $\psi' > 0$ must hold at all times, i.e. it is impossible to integrate for any $\tau_o < \tau_H$. Nevertheless, it is possible to obtain the formal equation for ψ upon the supposition the initial condition for (5.6) will not be $\psi(0) = 0$. Hence, taking an initial point $\tau_{\mathrm{i}} \geqslant \tau_H$, a final point $\tau_{\mathrm{f}} > \tau_{\mathrm{i}}$ and $\tau_o = \tau_{\mathrm{i}} + \tau$ with $\tau \in [0, \tau_{\mathrm{f}} - \tau_{\mathrm{i}}]$, a simple integration yields

$$\int_0^{\tau_{\mathrm{f}}-\tau_{\mathrm{i}}} \psi'(\tau)d\tau = \psi\left(\tau_{\mathrm{f}} - \tau_{\mathrm{i}}\right) - \psi(0) = \int_0^{\tau_{\mathrm{f}}-\tau_{\mathrm{i}}} \lambda^{(0)} d\tau = \int_{\tau_{\mathrm{i}}}^{\tau_{\mathrm{f}}} \lambda^{(0)} d\tau_o = \int_{\lambda_{\mathrm{i}}^{(0)}}^{\lambda_{\mathrm{f}}^{(0)}} \lambda^{(0)}\, f'\left(\lambda^{(0)}\right)\, d\lambda^{(0)}\,, \tag{5.7}$$

where

$$f\left(\lambda^{(0)};\gamma\right) = \tau_o\left(\lambda^{(0)}\right) = \left\{\left(\frac{m^2 q}{2\left(\mathfrak{m} - \lambda^{(0)\gamma}\right)} - s\right)^2 - 1\right\}\lambda^{(0)-\gamma}. \tag{5.8}$$

Integrating by parts,

$$\begin{aligned} \psi\left(\tau_{\mathrm{f}} - \tau_{\mathrm{i}}\right) - \psi(0) &= \lambda_{\mathrm{f}}^{(0)} f\left(\lambda_{\mathrm{f}}^{(0)};\gamma\right) - \lambda_{\mathrm{i}}^{(0)} f\left(\lambda_{\mathrm{i}}^{(0)};\gamma\right) - \left(F\left(\lambda_{\mathrm{f}}^{(0)};\gamma\right) - F\left(\lambda_{\mathrm{i}}^{(0)};\gamma\right)\right), \\ F\left(\lambda^{(0)};\gamma\right) &= \int f d\lambda^{(0)}. \end{aligned} \tag{5.9}$$

With regular diffusion the integration of f is fairly straightforward, yielding (with λ replacing $\lambda^{(0)}$ as the classic eigenvalue)

$$F(\lambda;1) = \frac{m^4 q^2}{4\mathfrak{m}}\,\frac{1}{\mathfrak{m} - \lambda} + \left(\left(\frac{m^2 q}{2\mathfrak{m}} - s\right)^2 - 1\right)\ln\lambda - \frac{m^2 q}{\mathfrak{m}}\left(\frac{m^2 q}{4\mathfrak{m}} - s\right)\ln\left(\mathfrak{m} - \lambda\right). \tag{5.10}$$

For the particular case $(p, q, m, s) = (3, 2, 3, 0)$, the result was formerly obtained [18].

To generalise (5.10) for the anomalous case some ingenuity must be summoned. The next two sections address the computation of $F\left(\lambda^{(0)};\gamma\right)$ and the limit $\gamma \longrightarrow 1^-$.

5.2 Computation of F for $0 < \gamma < 1$

Rearranging (5.8) into a form expedient for integration,

$$f\left(\lambda^{(0)}\right) = \frac{m^4q^2}{4\mathfrak{m}}\,\frac{1}{\left(\mathfrak{m}-\lambda^{(0)\gamma}\right)^2} + \frac{m^2q}{\mathfrak{m}}\left(\frac{m^2q}{4\mathfrak{m}}-s\right)\frac{1}{\mathfrak{m}-\lambda^{(0)\gamma}} + \left(\left(\frac{m^2q}{2\mathfrak{m}}-s\right)^2-1\right)\lambda^{(0)-\gamma}. \quad (5.11)$$

Hence the computation of F involves two integrals

$$I_1 = \int \frac{d\lambda^{(0)}}{\mathfrak{m}-\lambda^{(0)\gamma}} \quad \text{and} \quad I_2 = \int \frac{d\lambda^{(0)}}{\left(\mathfrak{m}-\lambda^{(0)\gamma}\right)^2}. \quad (5.12)$$

To compute I_1 change the integration variable according to $\lambda^{(0)\gamma} = \mathfrak{m}\mathfrak{t}$ to obtain

$$I_1 = \frac{\mathfrak{m}^{\frac{1}{\gamma}-1}}{\gamma}\int \frac{\mathfrak{t}^{\frac{1}{\gamma}-1}}{1-\mathfrak{t}}\,d\mathfrak{t}, \qquad 0 < \mathfrak{t} < 1\,. \quad (5.13)$$

The interval $0 < \mathfrak{t} < 1$ ensues by the virtue of all relevant eigenvalues lying in the right half plane, but to the left of $\mathfrak{m}$, the eigenvalue of the local operator, and is essential for proper convergence. Then,

$$\int \frac{\mathfrak{t}^{\frac{1}{\gamma}-1}}{1-\mathfrak{t}}\,d\mathfrak{t} = \int \mathfrak{t}^{\frac{1}{\gamma}}\left(\frac{1}{\mathfrak{t}}+\frac{1}{1-\mathfrak{t}}\right)d\mathfrak{t} = \frac{\mathfrak{t}^{\frac{1}{\gamma}}}{\frac{1}{\gamma}} + \int \mathfrak{t}^{\frac{1}{\gamma}+1}\left(\frac{1}{\mathfrak{t}}+\frac{1}{1-\mathfrak{t}}\right)d\mathfrak{t} = \mathfrak{t}^{\frac{1}{\gamma}}\sum_{j=0}^{n}\frac{\mathfrak{t}^j}{\frac{1}{\gamma}+j} + \int \frac{\mathfrak{t}^{\frac{1}{\gamma}+n}}{1-\mathfrak{t}}\,d\mathfrak{t}.$$

Since $0 < \mathfrak{t} < 1$, the limit $n \longrightarrow \infty$ exists and

$$\int \frac{\mathfrak{t}^{\frac{1}{\gamma}-1}}{1-\mathfrak{t}}\,d\mathfrak{t} = \gamma\,\mathfrak{t}^{\frac{1}{\gamma}}\sum_{j=0}^{\infty}\frac{\frac{1}{\gamma}\mathfrak{t}^j}{\frac{1}{\gamma}+j} = \gamma\,\mathfrak{t}^{\frac{1}{\gamma}}\;{}_2\mathscr{F}_1\left(\frac{1}{\gamma},1;\frac{1}{\gamma}+1;\mathfrak{t}\right), \quad (5.14)$$

where ${}_2\mathscr{F}_1\left(a,b;c;\mathfrak{t}\right)$ is Gauß's hypergeometric function defined by

$${}_2\mathscr{F}_1\left(a,b;c;\mathfrak{t}\right) = \sum_{n=0}^{\infty}\frac{a^{(n)}b^{(n)}}{c^{(n)}\,n!}\,\mathfrak{t}^n, \qquad \kappa^{(n)} = \kappa(\kappa+1)\cdot\;\ldots\;\cdot(\kappa+n-1), \quad \kappa^{(0)} = 1. \quad (5.15)$$

Hypergeometric functions have been known to appear in similar contexts [21]. Analogously

$$I_2 = \frac{\mathfrak{m}^{\frac{1}{\gamma}-2}}{\gamma}\int \frac{\mathfrak{t}^{\frac{1}{\gamma}-1}}{(1-\mathfrak{t})^2}\,d\mathfrak{t}, \qquad 0 < \mathfrak{t} < 1\,, \quad (5.16)$$

and

$$\int \frac{\mathfrak{t}^{\frac{1}{\gamma}-1}}{(1-\mathfrak{t})^2}\,d\mathfrak{t} = \mathfrak{t}^{\frac{1}{\gamma}-1}\left(\frac{1}{1-\mathfrak{t}}-1\right) - (1-\gamma)\mathfrak{t}^{\frac{1}{\gamma}}\;{}_2\mathscr{F}_1\left(\frac{1}{\gamma},1;\frac{1}{\gamma}+1;\mathfrak{t}\right). \quad (5.17)$$

Combining I_1 and I_2,

$$F\left(\lambda^{(0)};\gamma\right) = \frac{m^4 q^2 {\lambda^{(0)}}^{1-\gamma}}{4\gamma \mathfrak{m}^2}\left(\frac{\mathfrak{m}}{\mathfrak{m}-\lambda}-1\right) + \left(\left(\frac{m^2 q}{2\mathfrak{m}}-s\right)^2-1\right)\frac{{\lambda^{(0)}}^{1-\gamma}}{1-\gamma}$$

$$+\frac{m^2 q}{\mathfrak{m}^2}\left(\frac{m^2 q}{4\mathfrak{m}}\left(2-\frac{1}{\gamma}\right)-s\right)\lambda^{(0)}\ {}_2\mathscr{F}_1\left(\frac{1}{\gamma},1;\frac{1}{\gamma}+1;\frac{\lambda^{(0)}}{\mathfrak{m}}\right). \tag{5.18}$$

Expression (5.18) was intentionally retained in its not fully simplified form for a more convenient comparison with the case of regular diffusion, where $\gamma = 1$.

5.2.1 *Limit* $\lim\limits_{\gamma\longrightarrow 1^-} F\left(\lambda^{(0)};\gamma\right)$

The comparison of (5.18) and (5.10) is not immediate. It is readily discerned that upon taking $\gamma = 1$ the first term in (5.18) matches that in (5.10) up to an additive constant. The functional form of the second term in (5.18), i.e. ${\lambda^{(0)}}^{1-\gamma}/(1-\gamma)$ corresponds to $\ln\lambda$ in (5.10). In this sense, (5.10) is not a proper limit of (5.18) as $\gamma \longrightarrow 1^-$: if $\gamma = 1$ is set prior to the integration, one obtains the logarithm, whereas at $\gamma < 1$ the fractional power ensues. To compare the third term, set $\gamma = 1$, $\lambda^{(0)} \longmapsto \lambda$ and observe that (for instance by a Taylor expansion)

$${}_2\mathscr{F}_1\left(1,1;2;\mathfrak{t}\right) = -\frac{1}{\mathfrak{t}}\ln(1-\mathfrak{t}), \tag{5.19}$$

so again the terms match up to an additive constant. Since the integrals I_1 and I_2 were indefinite and a transformation of variables was performed, the presence of these additional constants is to be expected and is of no importance, as (5.9) involves a difference of two values of F evaluated at distinct points τ_{f} and τ_{i}.

Normal diffusion is a special limit, often singular, of a whole family of processes, and was historically favoured because of the mathematical simplicity of its analysis. The above comparison facilitates the distinction between the regular and singular parts of this limit. The power law ${\lambda^{(0)}}^{1-\gamma}$ becomes a constant when $\gamma = 1$. Consequently, the dependence ${\lambda^{(0)}}^{1-\gamma}/(\mathfrak{m}-\lambda)$ becomes purely hyperbolic. In another instance, ostensibly the same power law ${\lambda^{(0)}}^{1-\gamma}/(1-\gamma)$ turns into a logarithmic growth of the disturbance amplitude with normal diffusion.

6 Conclusion

The Gierer–Meinhardt model was endowed with a memory operator in the form of a fractional time derivative acting on the concentration of both activator and inhibitor species. The modification entails a spike solution in an asymptotic regime, where the ratio of the reagents' diffusivities is of order $\mathcal{O}\left(\epsilon^{2\gamma}\right)$, $0 < \gamma < 1$, an improvement by comparison to the classic $\mathcal{O}\left(\epsilon^2\right)$ known to be unrealistic.

The spike on an infinite line was shown to retain its normal shape with exponentially decaying tails. In a stationary case the only effect of anomaly is manifested through the inner layer variable relating to the breadth of the spike, whereas with drift the influence of anomaly is seen in the drift time scale as well as an asymmetry of leftward and rightward motion.

The spike stability was compared with the normal case for a class of disturbances evolving exponentially in time to leading order. To this end an anomalous eigenvalue–eigenfunction pair was introduced and a non-local eigenvalue problem was derived. Due to a peculiarity of the fractional derivative operator it is possible to trace the eigenvalues only in the right half of the complex plane. The anomalous eigenvalue problem can be obtained from the regular concominant through the mapping $\lambda \longmapsto \lambda^{(0)^\gamma}$, permitting the construction of an anomalous eigenvalue trajectory by extracting the part of the normal trajectory satisfying $|\arg\lambda| \leqslant \frac{1}{2}\pi\gamma$ and mapping it by the $1/\gamma$ root transformation.

For the special case, where the kinetic exponents satisfy $p = 2m - 3$, the eigenvalue problem is explicitly solvable, similarly to the system with regular diffusion. The Hopf bifurcation frequency λ_H and threshold τ_H were obtained for the case $s = 0$, extending former results obtained for a particular set of (p, m) with normal diffusion to arbitrary (p, m) as well as γ. Asymptotic solutions were constructed near the representative values $\gamma = 0$ (strongest anomaly), $\gamma = \frac{1}{2}$ (mid-range anomalous value) and $\gamma = 1$ (regular diffusion), and the extent of their accuracy explored by juxtaposition to the full numerical solution. Parameter regimes, where the asymptotic solution captured the true solution exceptionally well, were outlined. The case $s > 0$ was found to be qualitatively similar.

A bifurcation delay was analysed through a WKB type of disturbance at the onset of Hopf instability. Again, former results derived for a particular set of kinetic exponents and regular diffusion were extended here to any quadruple (p, q, m, s) and anomaly index γ.

The sub-diffusive Gierer–Meinhardt system is a phenomenological model that facilitates the study of the effect of memory on spike patterns. Since the memory operator does not possess a proper limit at $\gamma \longrightarrow 1^-$, this conceptual discontinuity transcends into some of the expressions generalising the results known for simple diffusion. The dearth of proper limit at $\gamma \longrightarrow 1^-$ is a manifestation of the singularity of the simple diffusion as a limit of a family of continuous processes.

Full numerical simulation of system (3.1) is beyond the scope of this paper. The kernel singularity of the fractional operator (2.1) in conjunction with the slow algebraic decay makes such a simulation highly non-trivial. This might be an interesting topic for future research. In addition, observation of numerical behaviour in the stable regime ($\tau < \tau_H$) might furnish the furtherance of an analytical description therein. Another line of enquiry worthy of attention is an extension of the stability theory to classes of perturbations other than exponential and designation of the most unstable type of disturbance for the pertinent type of memory operator.

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