Bounding the Size of Graphs without $K_m$-minors.

Let $\varepsilon(G)$ be the number of edges in a graph $G$ and let $\nu(G)$ denote the number of vertices. For a vertex $v$ in $G$, let $d_G(v)$ denote the number of edges incident with $v$, which is called the degree of $v$. We let $N_G(v)$ denote the neighbour set of $v$; that is, it is the set of vertices which are joined by an edge to the vertex $v$ in $G$. Using a simple counting argument, one can show that

0.1 Lemma
\[ \sum_{v \in V(G)} d_G(v) = 2\varepsilon(G). \]

A subgraph $H$ of a graph $G$ is graph obtained from $G$ by first choosing the vertices of $H$ to be some subset, say $X$, of the vertices of $G$. Secondly, one chooses the edges of $H$ to be some subset of the edges $G$ which join vertices in $X$. If one choose all the edges in $G$ which connect vertices in $X$, then we say that $H$ is the subgraph induced by $X$. Let $K_m$ denote the complete graph on $m$ vertices. We have the following theorem of Mader:

0.2 Theorem (Mader)
Let $G$ be a simple graph. Then for all $m = 2, 3, \ldots$, if $G$ has no $K_m$-minor, then $\varepsilon(B) \leq (2^m - 1)\nu(G)$.

Proof Suppose that the theorem is false. Let $G$ be a minimum counterexample; that is, a counterexample to the theorem which has the fewest number of vertices. We have $\varepsilon(G) > (2^m - 1)\nu(G)$. By our choice, the theorem is true for all simple graphs with fewer than $\nu(G)$ vertices. Pick a vertex $v$ in $G$ where $d_G(v) \geq 1$. Let $H$ denote the subgraph of $G$ induced by the neighbour set of $G$, $N_G(v)$. Since $G$ has no $K_m$-minor, it follows that $H$ has no $K_{m-1}$-minor. Furthermore, since $H$ has fewer vertices than $G$, the theorem is true for $H$ and therefore, $\varepsilon(H) \leq (2^{m-1} - 1)\nu(H) \leq (2^{m-1} - 1)(\nu(G) - 1)$. By the Lemma 0.1, $2\varepsilon(H) = \sum_{v \in V(H)} d_H(v)$. Thus $\sum_{v \in V(H)} d_H(v) \leq 2 \cdot (2^{m-1} - 1)\nu(H)$. Consequently, the average degree in $H$ is at most $2^m - 2$. We can therefore pick a vertex $w \in V(H)$ where $d_H(w) \leq 2^m - 2$. Let $e = vw$ be the edge joining $v$ and $w$. Let $G'$ be the graph obtained by contracting $e$. By contracting $e$, we may create multiple edges, i.e. pairs of edges joining the same vertices. This means that $G'$ is possible not simple. In fact, the number of multiple edges created is exactly $d_H(w)$. Why? Because each neighbour $w'$ of $w$ in $H$ is joined to both $w$ and $v$ by edges, and hence belongs to a triangle containing $e$. Let $G''$ be the graph obtained from $G'$ by deleting one edge from each pair of multiple edges. Then $G''$ is simple and

$\varepsilon(G'') = \varepsilon(G') - d_H(w) = \varepsilon(G) - 1 - d_H(w) > (2^m - 1)\nu(G) - 1 - (2^m - 2) = (2^m - 1)(\nu(G) - 1) = (2^m - 1)\nu(G'').$
Thus $\varepsilon(G'') > (2^m - 1)\nu(G'')$, and $G''$ has no $K_m$-minor because, $G$ has no $K_m$-minor. This contradicts our assumption that $G$ was a minimum counterexample. Thus there can be no counterexamples and hence the theorem is true.

Let $h_m(n)$ denote the maximum number of edges a simple graph with $n$ vertices and no $K_m$-minor can have. We know the following:

- $h_m(n) \geq (m - 2)n - \binom{m-1}{2}$.
- equality holds for $m = 3, \ldots, 7$.

**0.3 Theorem**

Let $G$ be a simple graph with no $K_4$-minor. Then $\varepsilon(G) \leq 2\nu(G) - 3$.

**Proof**  We follow that same proof as for Theorem 0.2, choosing a minimum counterexample, etc. Notice that since $H$ has no $K_3$-minor, and it can have no cycles (any cycle can be contracted to form a triangle). Such a graph must have a vertex of degree 1. So we may pick $w \in V(H)$ such that $d_H(w) = 1$. Now the edge $e = vw$ belongs to exactly one triangle of $G$. Let $G'$ and $G''$ be as before. Then

$$
\varepsilon(G'') = \varepsilon(G') - d_H(w) = \varepsilon(G) - 2 > 2\nu(G) - 5 = 2(\nu(G) - 1) - 3 = 2\nu(G'') - 3.
$$

Then $G''$ would be a counterexample to the theorem, a contradiction.

**0.4 Question**

Is it true that if $G$ is a simple graph having no $K_m$-minor, then $G$ contains an edge belonging to at most $2^{m-1}$ triangles? What about $m$ triangles?

**0.5 Question**

Is it true that if $M$ is a simple binary matroid with no $K_4$-minor, then there is an element which belongs to at most three triangles?

**0.6 Question**

Is it true that if $M$ is a simple binary matroid with no $K_5$-minor, then there is an element which belongs to at most three triangles?

- To answer the first question, maybe we could refine the proof of Mader’s theorem to yield an answer.
- To answer the second question, maybe one might try and find out if there is a “structure” theorem for binary matroids not having a $K_4$-minor. Knowing the structure of these matroids may enable one to find an element which belongs to at most two triangles.
Answering the third question, is anybody’s guess! I suggest trying to find empirical evidence. For example, suppose we just generate a bunch of matroids with no $K_5$-minor at random, and then test whether each contains an element belonging to at most 3 triangles. Either find a counterexample, or bolster the case for it being correct.