1 Parametric Bessel Equation and Bessel-Fourier Series

Recall the parametric Bessel equation of order \( n \):

\[
x^2 y'' + xy' + (\alpha^2 x^2 - n^2)y = 0
\]  

(1.1)

The general solution is given by \( y = J_n(\alpha x) + Y_n(\alpha x) \). If we put equation (1.1) into self-adjoint form we get

\[
xy'' + y' + (\alpha^2 x - \frac{n^2}{x})y = 0
\]  

(1.2)

Compare this with the self-adjoint form of the general two-point equation:

\[
r(x)y'' + r'(x)y' + (q(x) + \lambda p(x))y = 0
\]  

(1.3)

Comparing with equation (1.2), we have

\[
\lambda = \alpha^2, \quad r(x) = x, \quad p(x) = x, \quad q(x) = -\frac{n^2}{x}.
\]

We note that \( r(0) = 0 \) and the solutions \( J_n(\alpha x) \) are bounded at \( x = 0 \) whereas \( Y_n(\alpha x) \) is not. Now equation (1.2) together with \( r(0) = 0 \) and the boundary condition

\[
Ay(b) + By'(b) = 0
\]  

(1.4)

form a singular 2-point boundary-value problem. Now there are solutions to this problem for distinct values of \( \alpha \), say \( \alpha_1 < \alpha_2 < \alpha_3 < \cdots \) and the corresponding solutions are given by \( y = J_n(\alpha_i x) \), \( i = 1, 2, 3, \ldots \) (the eigenfunctions). Also these solutions are orthogonal with respect to the weighted inner product \( (f, g) = \int_0^b p(x)f(x)g(x) \, dx = \int_0^b xf(x)(g) \, dx \). Thus we have that

\[
\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x) \, dx = 0, \text{ for all } i \neq j
\]  

(1.5)

2 Fourier-Bessel Series

Our goal is to show that Bessel functions can play the role of sine and cosine functions in a Fourier series, which we call Fourier-Bessel series. We shall determine some of these series explicitly in special cases.

Recall that if \( \phi_1(x), \phi_2(x), \phi_3(x), \ldots \) are orthogonal functions on an interval \([a,b] \), and \( f(x) \) is a piecewise continuous function on \([a,b] \) where \( f(x) = \sum_{i=1}^{\infty} c_i \phi_i(x) \), then the coefficients \( c_i \) are given by

\[
c_i = \frac{\langle \phi_i, f \rangle}{\|\phi_i\|^2}, \quad i = 1, 2, 3, \ldots
\]  

(2.1)
Now the functions \( J_n(\alpha x) \), \( i = 1, 2, 3, \ldots \) are orthogonal (see (1.5)). Moreover, the \( \alpha \)'s are determined by condition (1.4) which in this case means

\[
AJ_n(\alpha b) + B\alpha J'_n(\alpha b) = 0, \quad i = 1, 2, 3, \ldots
\]  
(2.2)

Now if for a piecewise-continuous function \( f(x) \) on the interval \([0, b]\) we have \( f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \), then the coefficients are given by

\[
c_i = \frac{(f(x), J_n(\alpha_i x))}{\|J_n(\alpha_i x)\|^2} = \frac{\int_{0}^{b} x f(x) J_n(\alpha_i x) \, dx}{\int_{0}^{b} x J_n^2(\alpha_i x) \, dx}, \quad i = 1, 2, 3, \ldots
\]  
(2.3)

3 Determining the Coefficients \( c_i \)

We start by recalling some facts about Bessel functions. The (first order) Bessel function of order \( n \) is given by the series \( \sum_{n=0}^{\infty} \frac{(-1)^j}{j!\Gamma(1+n+i)} \left( \frac{x}{2} \right)^{2j+n} \). From this series we can derive the following:

\[
\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x)
\]  
(3.1)

\[
\frac{d}{dx} [x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)
\]  
(3.2)

From (3.2), we obtain \(-nx^{-n-1} J_n(x) + x^{-n} J'_n(x) = -x^{-n} J_{n+1}(x)\). Multiplying this equation by \( x^{n+1} \) we obtain \(-n J_n(x) + x J'_n(x) = -x J_{n+1}(x)\). Thus we have the relation

\[
x J'_n(x) = n J_n(x) - x J_{n+1}(x).
\]  
(3.3)

We want to use the above equations to evaluate \( \|J_n(\alpha_i x)\|^2 \). The differential equation (1.2) can be put into the form

\[
\frac{d}{dx} [x y'] + (\alpha^2 x - \frac{n^2}{x}) y = 0
\]  
(3.4)

We multiply the above equation by \( 2xy' \) yielding

\[
2xy' \frac{d}{dx} [x y'] + 2xy' (\alpha^2 x - \frac{n^2}{x}) y = 0
\]

\[
\frac{d}{dx} [(x y')^2] + (\alpha^2 x^2 - n^2) \frac{d}{dx} [y^2] = 0
\]

\[
\int_{0}^{b} \frac{d}{dx} [(x y')^2] + (\alpha^2 x^2 - n^2) \frac{d}{dx} [y^2] \, dx = 0
\]

\[
(x y')^2 \big|_0^b + \int_{0}^{b} (\alpha^2 x^2 - n^2) \frac{d}{dx} [y^2] \, dx = 0
\]

\[
(x y')^2 \big|_0^b + (\alpha^2 x^2 - n^2) y \big|_0^b - \int_{0}^{b} 2\alpha^2 x y^2 \, dx = 0 \quad \text{(integrating by parts)}
\]

\[
\int_{0}^{b} 2\alpha^2 x y^2 \, dx = (xy')^2 \big|_0^b + (\alpha^2 x^2 - n^2) y^2 \big|_0^b.
\]
The above equation must hold for each } \alpha = \alpha_i, \ i = 1, 2, 3, \ldots \text{. Thus we have }

\[
\int_0^b 2 \alpha_i^2 y^2 \, dx = (xy_i')^2 \bigg|_0^b + (\alpha_i^2 x^2 - n^2)^2 \bigg|_0^b, \ i = 1, 2, 3, \ldots \quad (3.5)
\]

Let } y = J_n(\alpha x) \text{. Then } y' = \alpha x J_n'(\alpha x) \text{ (using the chain rule). We also know that } J_n(0) = 0 \text{ if } n > 0, \text{ and } J_0(0) = 1. \text{ By (3.3), we have that } xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x). \text{ So } xJ_n'(x) = 0 \text{ at } x = 0. \text{ Substituting } y = J_n(\alpha x) \text{ into (3.5), we get the following: }

\[
\int_0^b 2 \alpha_i^2 x J_n^2(\alpha x) \, dx = (\alpha_i b J_n'(\alpha b))^2 + (\alpha_i^2 b^2 - n^2) J_n^2(\alpha b) \quad (3.6)
\]

Substituting } x = \alpha b \text{ into (3.3), we get } \alpha b J_n'(\alpha b) = nJ_n(\alpha b) - \alpha b J_{n+1}(\alpha b). \text{ Substituting this into (3.6), we get }

\[
\int_0^b 2 \alpha_i^2 x J_n^2(\alpha x) \, dx = \frac{(nJ_n(\alpha b) - \alpha b J_{n+1}(\alpha b))^2 + (\alpha b^2 - n^2) J_n^2(\alpha b)}{\alpha_i^2 b^2 (J_n^2(\alpha b) + J_{n+1}^2(\alpha b)) - 2n\alpha b J_n(\alpha b)J_{n+1}(\alpha b).}
\]

So we obtain from the above that

\[
||J_n(\alpha x)||^2 = \int_0^b x J_n^2(\alpha x) \, dx = \frac{b^2}{\alpha_i^2} (J_n^2(\alpha b) + J_{n+1}^2(\alpha b)) - \frac{2nb}{\alpha_i} J_n(\alpha b)J_{n+1}(\alpha b). \quad (3.7)
\]

We shall look at some special cases where } ||J_n(\alpha)||^2 \text{ is simpler.}

Case 1 Suppose } A = 1 \text{ and } B = 0 \text{ in (1.4).}

In this case, the boundary condition (1.4) becomes } Ay(b) = 0, \text{ or } y(b) = 0. \text{ For } y = J_n(\alpha x), \text{ this translates into }

\[
J_n(\alpha b) = 0. \quad (3.8)
\]

Let } x_1, x_2, x_2, \ldots \text{ be the roots of function } J_n(x), \text{ that is where it is zero. Let } \alpha_i, \ i = 1, 2, 3, \ldots \text{ be the values of } \alpha \text{ satisfying (3.8). Then }

\[
J_n(\alpha_i b) = 0, \ \alpha_i b = x_i, \ \alpha_i = \frac{x_i}{b}, \ i = 1, 2, 3, \ldots .
\]

Since } J_n(\alpha_i b) = 0, \text{ equation (3.7) implies that }

\[
||J_n(\alpha x)||^2 = \frac{b^2}{\alpha_i^2} J_{n+1}^2(\alpha_i b), \ i = 1, 2, 3, \ldots \quad (3.9)
\]

So from (2.3), the coefficients are

\[
c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b xf(x)J_n(\alpha x) \, dx, \ i = 1, 2, 3, \ldots \quad (3.10)
\]

3
Case 2 Suppose \( A = h > 0 \) and \( B = b \) in (1.4).

In this case, the boundary condition (1.4) becomes \( hy(b) + by'(b) = 0 \). For \( y = J_\alpha(\alpha x) \), this translates into

\[
h J_\alpha(ab) + \alpha b J'_\alpha(ab) = 0.
\]

(3.11)

Let \( \alpha_1, \alpha_2, \alpha_3, \ldots \) be the values of \( \alpha \) which satisfy equation (3.11). Then we have that

\[
\alpha_i b J'_\alpha(ab) = -h J_\alpha(ab), \quad i = 1, 2, 3, \ldots
\]

Substituting the above into (3.6), we obtain that

\[
\int_0^b 2 \alpha_i^2 x J_n^2(\alpha_i x) \, dx = h^2 J_n^2(ab) + (\alpha_i^2 b^2 - n^2) J_n^2(ab)
\]

\[
= (h^2 + (\alpha_i^2 b^2 - n^2)) J_n^2(ab).
\]

So we get

\[
\|J_n(ab)\|^2 = \int_0^b x J_n^2(ab) \, dx = \frac{(h^2 + (\alpha_i^2 b^2 - n^2))}{2 \alpha_i^2} J_n^2(ab).
\]

(3.12)

By (2.3), the coefficients are:

\[
\alpha_i = \frac{2 \alpha_i^2}{(h^2 + (\alpha_i^2 b^2 - n^2)) J_n^2(ab)} \int_0^b x f(x) J_n(ab) \, dx, \quad i = 1, 2, 3, \ldots
\]

(3.13)

As special case of case 2 above we have

Case 3 \( n = 0, A = h = 0, \) and \( B = b \).

In this case, the boundary condition (3.11) becomes

\[
\alpha_i b J'_\alpha(ab) = 0, \quad J'_\alpha(ab) = 0.
\]

(3.14)

As before, let \( \alpha_i, \, i = 1, 2, 3, \ldots \) be the values of \( \alpha \) which satisfy (3.14). Then

\[
J'_\alpha(ab) = 0, \quad i = 1, 2, 3, \ldots
\]

(3.15)

Now by (3.2), we have \( J'_\alpha(x) = -J_1(x) \). Thus by (3.15), we have

\[
J_1(ab) = 0, \quad i = 1, 2, 3, \ldots
\]

(3.16)

In particular, if \( x_1 < x_2 < x_3 < \cdots \) are the zeros of \( J_1(x) \), then \( \alpha_i = \frac{x_i}{b}, \, i = 1, 2, 3, \ldots \). One difference with the previous cases is that since \( J'_0(0) = J_1(0) = 0 \) and \( J_0(0) = 1 \), we see that \( \alpha = 0 \) is a solution to (3.14), and \( y = J_0(0) = 1 \) is a nontrivial solution. So \( \lambda = 0 \) is an eigenvalue with corresponding eigenfunction \( y = 1 \). We note that \( \|1\|^2 = \int_0^b x \, dx = \frac{b^2}{2} \). To summarize, in this case, \( f(x) = c_0 + \sum_{i=1}^{\infty} c_i J_i(\alpha_i x) \) where

\[
c_0 = \frac{2}{b^2} \int_0^b x \, dx, \quad c_i = \frac{2}{b^2 J_i(ab)} \int_0^b x f(x) J_i(ab) \, dx, \quad i = 1, 2, 3, \ldots
\]

(3.17)
4 An Example

Suppose we want to find the Fourier-Bessel series for $f(x) = x^2$ on the interval $[0, 2]$ where the boundary condition is $J_0'(2\alpha) = 0$.

This corresponds to Case 3 above where $b = 2$. We need to find the coefficients $c_0, c_1, c_2, \ldots$ where $f(x) = x^2 = c_0 + \sum_{i=1}^{\infty} c_i J_0(\alpha_i x)$. These are given by (3.17). Here $c_0 = \frac{2}{2\pi} \int_0^2 x^2 \, dx = 2$. We also have $c_i = \frac{1}{2} \int_0^2 x^2 J_0(\alpha_i x) \, dx$. Let $t = \alpha_i x$. Then $dx = \frac{1}{\alpha_i} dt$, and $c_i = \frac{1}{2} \alpha_i^2 \int_0^2 t^2 \cdot t J_0(t) \, dt$. To evaluate the integral, we note that by (3.1), we have $\frac{d}{dt}[t^n J_n(t)] = t^n J_{n-1}(t)$. So for $n = 1$ we have $\frac{d}{dt}[t J_1(t)] = t J_0(t)$.

Substituting this into the integral we have: $\int_0^2 t^2 \cdot t J_0(t) \, dt = \int_0^2 t^2 J_0(t) \, dt - \int_0^2 2t \cdot t J_1(t) \, dt$. Integrating by parts we obtain

$$\int_0^{2\alpha_i} t^2 \frac{d}{dt}[t J_1(t)] \, dt = r^2 J_1(t)|^2_{2\alpha} - \int_0^{2\alpha_i} 2t \cdot t J_1(t) \, dt.$$

We also have $\frac{d}{dt}[t J_2(t)] = t J_1(t)$. Thus

$$\int_0^{2\alpha_i} 2t \cdot t J_1(t) \, dt = \int_0^{2\alpha_i} 2t \frac{d}{dt}[t J_2(t)] \, dt.$$

Continue by integrating by parts:

$$\int_0^{2\alpha_i} 2t \frac{d}{dt}[t J_2(t)] \, dt = 2t^2 J_2(t)|^{2\alpha_i} - 2 \int_0^{2\alpha_i} t J_2(t) \, dt = 2(2\alpha_i)^2 J_2(2\alpha_i) - 2 (2\alpha_i) J_2(2\alpha_i).$$

Now, put everything together to get $c_i = \ldots$