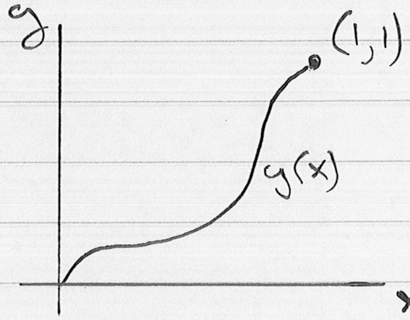
6.9



To make stationary

$$I[y] = \int_0^1 \underbrace{\left(y'^2 + yy' + y^2\right)}_{F(y,\,y')} dx$$

use Euler-Lagrange:

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'} \rightarrow y' + 2y = \frac{d}{dx}\left(2y' + y\right)$$

$$= 2y'' + y'$$

$$\rightarrow y'' - y = 0 \quad (\text{a } 2^{nd}\text{-order linear ODE})$$

assume $y(x) = e^{rx} \rightarrow r^2 - 1 = 0 \rightarrow r = \pm 1$

$\therefore\ y(x) = Ae^{x} + Be^{-x}$

impose boundary conditions:

$$\begin{cases} 0 = y(0) = A + B \rightarrow B = -A \\ 1 = y(1) = Ae + Be^{-1} \rightarrow 1 = Ae - Ae^{-1} \end{cases}$$

$$\rightarrow A = \frac{1}{e - e^{-1}}$$

$$\therefore \boxed{y(x) = \frac{e^{x} - e^{-x}}{e - e^{-1}} = \frac{\sinh(x)}{\sinh(1)}}$$