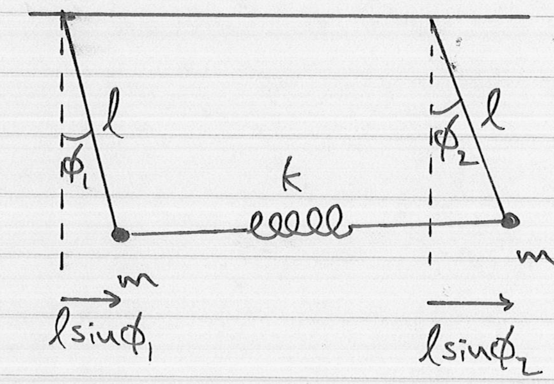
11.14



Spring at rest length when $\phi_1 = \phi_2 = 0$.

$$T = \frac{1}{2}m(l\dot{\phi}_1)^2 + \frac{1}{2}m(l\dot{\phi}_2)^2$$

For small angles, deflection of spring from horizontal is negligible ... spring stretch is $l\sin\phi_2 - l\sin\phi_1$ ∴

$$\Rightarrow U \approx \frac{1}{2}k(l\underbrace{\sin\phi_2}_{\approx\phi_2} - l\underbrace{\sin\phi_1}_{\approx\phi_1})^2 - mgl\cos\phi_1 - mgl\cos\phi_2$$

$$\Rightarrow L = T - U$$
$$= \frac{1}{2}ml^2(\dot{\phi}_1^2 + \dot{\phi}_2^2) - \frac{1}{2}kl^2(\phi_2 - \phi_1)^2 + mgl(\cos\phi_1 + \cos\phi_2)$$

Equations of motion:

$$\frac{\partial L}{\partial \phi_1} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_1} \Rightarrow ml^2\ddot{\phi}_1 = kl^2(\phi_2 - \phi_1) - mgl\underbrace{\sin\phi_1}_{\approx\phi_1}$$

$$\Rightarrow ml\ddot{\phi}_1 = kl(\phi_2 - \phi_1) - mg\phi_1 \qquad (1)$$

$$\frac{\partial L}{\partial \phi_2} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_2} \Rightarrow ml^2\ddot{\phi}_2 = -kl^2(\phi_2 - \phi_1) - mgl\underbrace{\sin\phi_2}_{\approx\phi_2}$$

$$\Rightarrow ml\ddot{\phi}_2 = -kl(\phi_2 - \phi_1) - mg\phi_2 \qquad (2)$$

In matrix form:

$$\underbrace{\begin{bmatrix} ml & 0 \\ 0 & ml \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix}}_{\ddot{\underline{x}}} = \underbrace{\begin{bmatrix} -kl-mg & kl \\ kl & -kl-mg \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}}_{\underline{x}}$$

Assume $\underline{x}(t) = \underline{z}\, e^{i\omega t} \Rightarrow (K + \omega^2 M)\underline{z} = 0$

with:

$$K + \omega^2 M = \begin{bmatrix} \omega^2 ml - kl - mg & kl \\ kl & \omega^2 ml - kl - mg \end{bmatrix}$$

To get non-trivial solutions we need:

$$0 = \det(K + \omega^2 M) = (\omega^2 ml - kl - mg)^2 - (kl)^2$$

$$\Rightarrow \omega^2 ml - kl - mg = \pm kl$$

$$\Rightarrow \omega^2 = \frac{mg + kl \pm kl}{ml} = \frac{g}{l} \text{ or } \frac{g}{l} + \frac{2k}{m}$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{g}{l}} \text{ or } \sqrt{\frac{g}{l} + 2\frac{k}{m}}}$$

Now solve for $\underline{z}$ to get modes...

case $\omega^2 = \frac{g}{l}$:

$$\underline{0} = \left(K + \frac{g}{l} M\right)\underline{z} = \begin{bmatrix} -kl & kl \\ kl & -kl \end{bmatrix} \underline{z} \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \boxed{\underline{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Motion is in-phase. ω is indep. of k since spring remains at rest length.

case $\omega^2 = \frac{g}{l} + 2\frac{k}{m}$:

$$\underline{0} = \left(K + \left[\frac{g}{l} + 2\frac{k}{m}\right] M\right)\underline{z} = \begin{bmatrix} kl & kl \\ kl & kl \end{bmatrix} \underline{z} \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \boxed{\underline{z} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Motion is out-of-phase.

mode 1: $\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_1 t}$

mode 2: $\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\omega_2 t}$