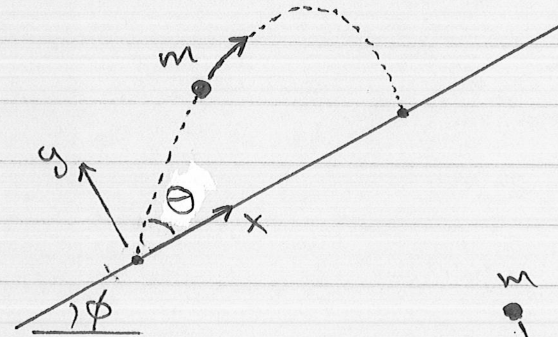
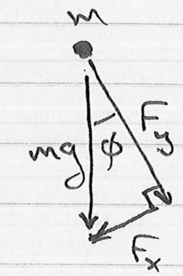
1.39



2^{nd} law:

$$m\ddot{x} = F_x = -mg\sin\phi$$
$$m\ddot{y} = F_y = -mg\cos\phi$$



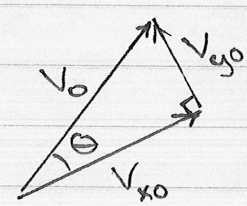
$$\rightarrow \begin{cases} \ddot{x} = -g\sin\phi \\ \ddot{y} = -g\cos\phi \end{cases}$$

integrate twice to get kinematics formulas:

$$\begin{cases} x(t) = x_0 + v_{x0}t - \frac{1}{2}g\sin\phi \cdot t^2 \\ y(t) = y_0 + v_{y0}t - \frac{1}{2}g\cos\phi \cdot t^2 \end{cases}$$

- with origin at initial position:

$$x_0 = y_0 = 0$$
$$v_{x0} = v_0\cos\theta, \quad v_{y0} = v_0\sin\theta$$



$$\rightarrow \boxed{\begin{cases} x(t) = v_0\cos\theta \cdot t - \frac{1}{2}g\sin\phi \cdot t^2 \\ y(t) = v_0\sin\theta \cdot t - \frac{1}{2}g\cos\phi \cdot t^2 \end{cases}}$$

To find landing point:

$$0 = y(t_*) = t_*\left(v_0\sin\theta - \frac{1}{2}g\cos\phi \cdot t_*\right)$$

$$\Rightarrow t_* = 0 \quad \text{or} \quad t_* = \frac{2v_0\sin\theta}{g\cos\phi}$$

$\therefore$ landing point is at:

$$R = x(t_*) = \left(v_0 \cos\theta - \frac{1}{2} g \sin\phi \cdot \frac{2 v_0 \sin\theta}{g \cos\phi} \right) \frac{2 v_0 \sin\theta}{g \cos\phi}$$

$$= v_0 \left(\cos\theta - \frac{\sin\phi \sin\theta}{\cos\phi} \right) \frac{2 v_0 \sin\theta}{g \cos\phi}$$

$$= \frac{2 v_0^2}{g} \left(\frac{\cos\theta \cos\phi - \sin\phi \sin\theta}{\cos\phi} \right) \frac{\sin\theta}{\cos\phi}$$

$$\rightarrow \quad \boxed{R = \frac{2 v_0^2}{g} \cdot \frac{\cos(\theta + \phi) \sin\theta}{\cos^2\phi}}$$

To choose θ to maximize R:

$$0 = \frac{dR}{d\theta} = \frac{2 v_0^2}{g \cos^2\phi} \cdot \left[\cos\theta \cdot \cos(\theta + \phi) - \sin\theta \cdot \sin(\theta + \phi) \right]$$

$$= \frac{2 v_0^2}{g \cos^2\phi} \cdot \cos(2\theta + \phi)$$

$$\rightarrow \quad 2\theta + \phi = \frac{\pi}{2} + k\pi \quad (k = 0, \pm 1, \pm 2, \ldots)$$

$$\rightarrow \quad \theta = \frac{1}{2} \left(\frac{\pi}{2} - \phi + k\pi \right)$$

$$= \frac{\pi}{4} - \frac{\phi}{2} + k\frac{\pi}{2} \qquad \left(k \neq 0 \text{ gives } \theta \text{ outside } \left[0, \tfrac{\pi}{2}\right] \right)$$

$$= \frac{\pi}{4} - \frac{\phi}{2}$$

$$\rightarrow \cancel{2\theta + \phi = 2\left(\frac{\pi}{4} - \frac{\phi}{2}\right) + \phi = \frac{\pi}{2}}$$

$$\rightarrow \cancel{R_{max} = \frac{2 v_0^2}{g}}$$

$$\rightarrow R_{max} = \frac{2v_0^2}{g} \cdot \frac{\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}{\cos^2\phi}$$

note: $\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right) = \cos\frac{\pi}{4}\cos\frac{\phi}{2} - \sin\frac{\pi}{4}\sin\frac{\phi}{2}$

$$= \frac{1}{\sqrt{2}}\cos\frac{\phi}{2} - \frac{1}{\sqrt{2}}\sin\frac{\phi}{2}$$

$$\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right) = \sin\frac{\pi}{4}\cos\frac{\phi}{2} - \cos\frac{\pi}{4}\sin\frac{\phi}{2}$$

$$= \frac{1}{\sqrt{2}}\cos\frac{\phi}{2} - \frac{1}{\sqrt{2}}\sin\frac{\phi}{2}$$

$$\therefore \cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right) = \frac{1}{2}\left(\cos\frac{\phi}{2} - \sin\frac{\phi}{2}\right)^2$$

$$= \frac{1}{2}\left(\cos^2\frac{\phi}{2} - 2\cos\frac{\phi}{2}\sin\frac{\phi}{2} + \sin^2\frac{\phi}{2}\right)$$

$$= \frac{1}{2}\left(1 - 2\cos\frac{\phi}{2}\sin\frac{\phi}{2}\right)$$

$$= \frac{1}{2}\left(1 - \sin\phi\right) \quad \text{[double angle]}$$

So finally:

$$R_{max} = \frac{2v_0^2}{g} \cdot \frac{\frac{1}{2}(1-\sin\phi)}{\cos^2\phi}$$

$$= \frac{v_0^2}{g} \cdot \frac{1-\sin\phi}{1-\sin^2\phi} = \frac{v_0^2}{g} \cdot \frac{1-\sin\phi}{(1-\sin\phi)(1+\sin\phi)}$$

$$\rightarrow \boxed{R_{max} = \frac{v_0^2}{g}\frac{1}{1+\sin\phi}}$$