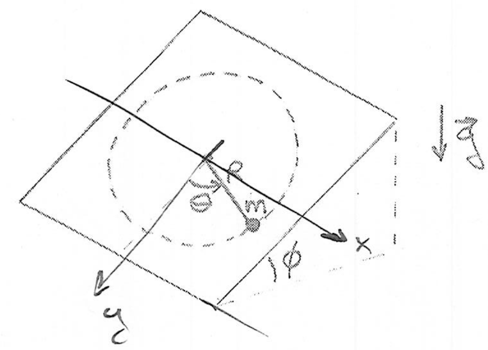
/12 **Problem 1:** Consider a simple pendulum consisting of a point mass m connected by a massless rigid rod (length R) to a central pivot. The pendulum assembly lies on a frictionless plane surface that is inclined at an angle ϕ ($0 \leq \phi \leq 90°$) above horizontal (see the diagram below).

(a) Write an expression for the Lagrangian of this system in terms of the coordinate θ.

/3

$T = \frac{1}{2}m(R\dot{\theta})^2 \qquad U = -mgy\sin\phi$

$= -mgR\cos\theta\sin\phi$



$\rightarrow L = T - U$

$= \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta\sin\phi$

(b) Show that the angular momentum is conserved for all motions if and only if $\phi = 0$. (or $\theta(t) = 0$ ... i.e. eqm)

/2

$\frac{\partial L}{\partial \theta} = \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)}_{\text{ang. momentum } mR^2\dot{\theta}} = -mgR\sin\theta\sin\phi \equiv 0$ iff $\sin\phi = 0$

$\Leftrightarrow \phi = 0$

(c) Find the equation of motion for θ. Show that the motion is identical to that of a vertical simple pendulum of some length l; find an expression for l in terms of ϕ.

/3

$\cancel{m}R^2\ddot{\theta} = -\cancel{m}gR\sin\theta\sin\phi \rightarrow \boxed{\ddot{\theta} + \left(\frac{g}{R}\sin\phi\right)\sin\theta = 0}$

compare: $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$ for vertical pend.

These are identical if $\frac{g}{l} = \frac{g}{R}\sin\phi$:

$\rightarrow \boxed{l = \frac{R}{\sin\phi}}$

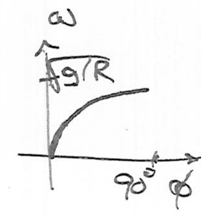
(d) Give an expression, in terms of ϕ, for the frequency of small oscillations about the equilibrium $\theta = 0$. Verify that your answer gives the expected result in the limiting case $\phi \to 90°$. What happens in the limiting case $\phi \to 0$?

/4

$|\theta| \ll 1 \Rightarrow \sin\theta \approx \theta$

$\Rightarrow \ddot{\theta} + \left(\frac{g}{R}\sin\phi\right)\theta = 0$

$\theta(t) = e^{i\omega t} \Rightarrow -\omega^2 + \frac{g}{R}\sin\phi = 0 \Rightarrow \boxed{\omega = \sqrt{\frac{g}{R}\sin\phi}}$



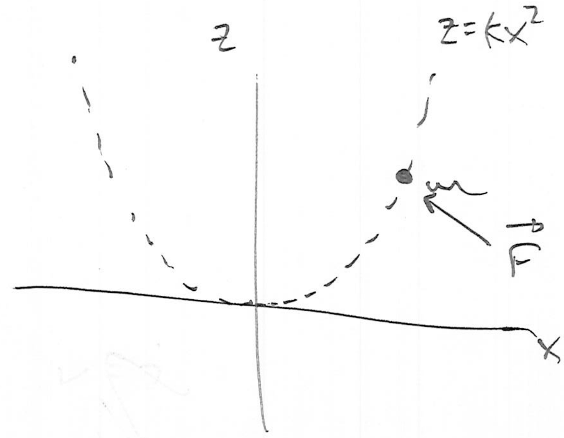
$\phi \to 90°$: $\omega \to \sqrt{\frac{g}{R}}$ as expected for vertical pendulum

$\phi \to 0°$: $\omega \to 0$ (period grows without bound)

$\phi = 0$: $\ddot{\theta} = 0 \Rightarrow \dot{\theta} = C \Rightarrow$ const. angular velocity

/10

Problem 2: A bead of mass m moves on a frictionless wire that is bent into the shape of the curve $z = kx^2$ in the vertical xz-plane. Find an expression, in terms of x and $\dot{x}$, for the force that the wire exerts on the bead. Verify that your expression gives the expected result in the case $x = \dot{x} = 0$.



$$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2)$$

$$U = mgz$$

$$\to L = T - U$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz$$

$$\underbrace{z - kx^2}_{f(x,z)} = 0$$

(1) $$\frac{\partial}{\partial x}(L + \lambda f) = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

$$\Rightarrow \underbrace{-\lambda 2kx}_{F_x \text{ due to wire}} = m\ddot{x}$$

(2) $$\frac{\partial}{\partial z}(L + \lambda f) = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}}$$

$$\Rightarrow -mg + \underbrace{\lambda}_{F_z \text{ due to wire}} = m\ddot{z}$$

(3) $$\dot{z} = 2kx\dot{x}$$

$$\Rightarrow \ddot{z} = 2k\dot{x}^2 + 2kx\ddot{x} \quad (3)$$

$$(3) \Rightarrow \left(\frac{\lambda}{m} - g\right) = 2k\dot{x}^2 + 2kx\left(\frac{-\lambda 2kx}{m}\right) \Rightarrow \lambda\left(\frac{1}{m} + \frac{4k^2x^2}{m}\right) = 2k\dot{x}^2 + g$$

$$\Rightarrow \lambda = \frac{m(2k\dot{x}^2 + g)}{1 + 4k^2x^2}$$

$$\therefore \vec{F} = (F_x, F_z)$$

$$= \lambda(-2kx, 1) = \cancel{-2kx\,m(2k\dot{x}^2+g)}$$

$$\Rightarrow \boxed{\vec{F} = \frac{m(2k\dot{x}^2 + g)}{1 + 4k^2x^2}(-2kx, 1)}$$

$\dot{x} = x = 0$

$$\Rightarrow \boxed{\vec{F} = (0, mg)}$$

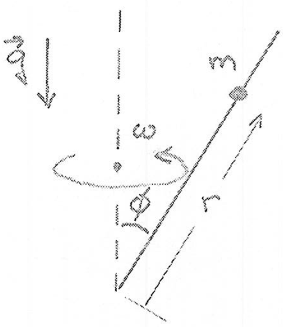
weight of m, as expected

/11

Problem 3: A bead of mass m moves along a frictionless straight wire. The wire is kept at a fixed angle ϕ away from vertical, and is forced to rotate at constant angular speed ω about a vertical axis passing through the lower end of the wire (see the diagram below).

(a) Find the equation of motion of the bead in terms of the coordinate r. Verify that your answer gives the expected result in the limiting case $\phi \to 0$.

/5



$$T = \frac{1}{2}m\left((\omega r \sin\phi)^2 + \dot{r}^2\right)$$

$$U = mgr\cos\phi$$

$$\Rightarrow L = T - U = \frac{1}{2}m\omega^2 r^2 \sin^2\phi + \frac{1}{2}m\dot{r}^2 - mgr\cos\phi$$

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \Rightarrow \underbrace{\cancel{m}\omega^2 r \sin^2\phi}_{\text{centrifugal}} \underbrace{- \cancel{m}g\cos\phi}_{\text{gravity}} = \cancel{m}\ddot{r}$$

$$\Rightarrow \boxed{\ddot{r} - (\omega^2 \sin^2\phi) r = -g\cos\phi}$$

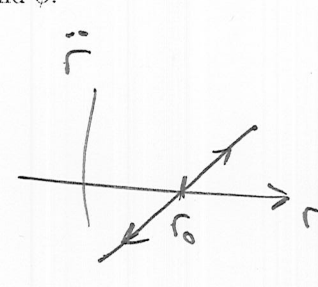
$\phi \to 0$: $\ddot{r} = -g$ ✓ (freefall under gravity)

(b) Show that there is just one equilibrium position $r = r_0$ for the bead. Find r_0 in terms of ω and ϕ.

/2

$$\ddot{r} = \dot{r} = 0 \Rightarrow -\omega^2 \sin^2\phi \, r_0 = -g\cos\phi$$

$$\Rightarrow \boxed{r_0 = \frac{g\cos\phi}{\omega^2 \sin^2\phi}}$$ is the only sol'n.



(c) Show that the bead does *not* exhibit oscillations about its equilibrium position. Solve the equation of motion and describe the bead's actual motion.

/4

2nd-order linear DE: $r(t) = Ae^{kt} + Be^{-kt} + C$

particular ~~homog.~~ soln: $C = r_0$

homog. soln: $k^2 - \omega^2 \sin^2\phi = 0 \Rightarrow k = \pm\omega\sin\phi$

$$\therefore\ r(t) = \underbrace{Ae^{(\omega\sin\phi)t} + Be^{-(\omega\sin\phi)t}}_{=0 \text{ only for some initial conditions;}} + \underbrace{\frac{g\cos\phi}{\omega^2\sin^2\phi}}_{\text{eq'm}}$$

otherwise $r(t)$ grows exponentially away from r_0