Mathematics and Music: Timbre and Consonance

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Introduction

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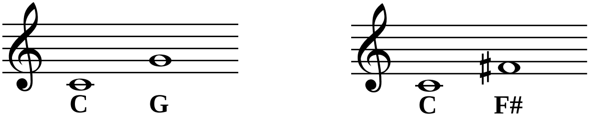
- Some musical intervals sound consonant (“good”?), others dissonant (“bad”?). Why?



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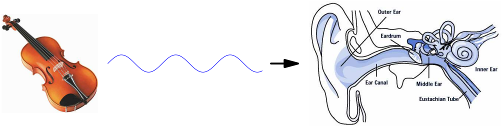

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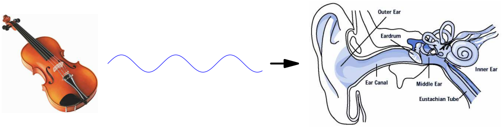
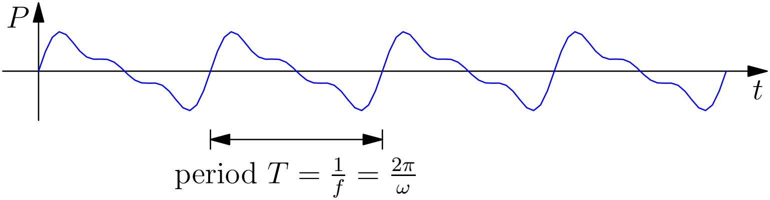


- Music is *organized sound*

Musical Tones

- **periodic** sound pressure is perceived as a musical tone:



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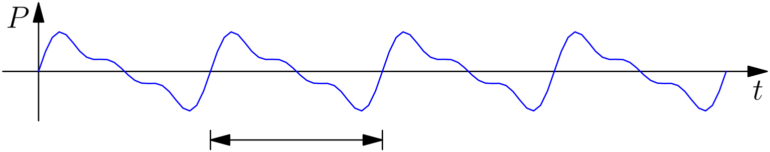

$$\text{period } T = \frac{1}{f} = \frac{2\pi}{\omega}$$

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higher $f \Leftrightarrow$ higher pitch

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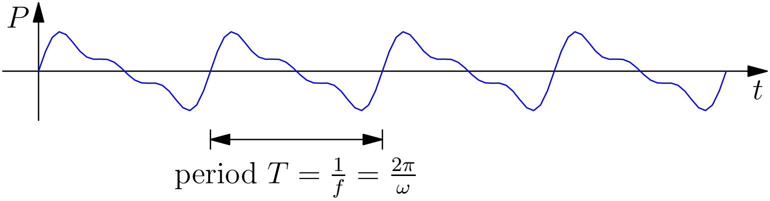

- the frequency f [Hz $=$ cycles/sec] is perceived as *pitch*

$$\text{higher } f \Leftrightarrow \text{higher pitch}$$

- a *pure tone* of frequency f is sinusoidal:

$$P(t) = A\sin(2\pi f t + \phi)$$

Frequency and Pitch

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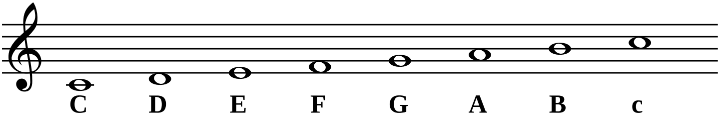
is a *geometric* sequence in frequency:

$$\{f_0, \alpha f_0, \alpha^2 f_0, \alpha^3 f_0, \ldots\}$$

C major scale (equal temperament)



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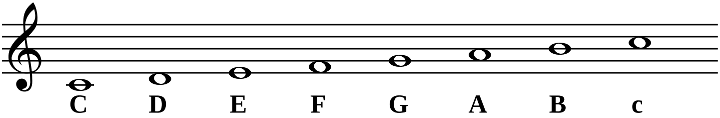

Subset of the 12-tone chromatic scale:

$$f_n = 440 \cdot 2^{n/12} \quad (n = -5, -4, \ldots, 2)$$

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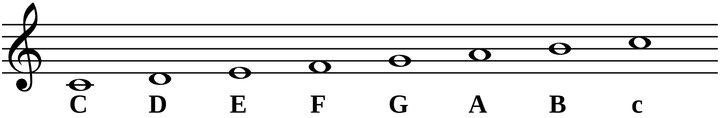
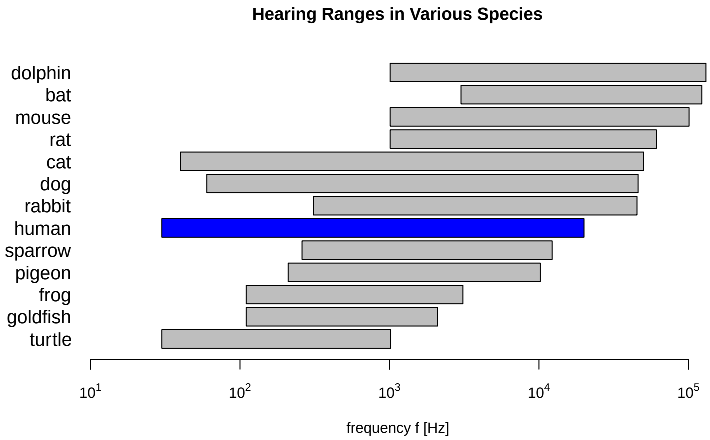


Subset of the 12-tone chromatic scale:

$$f_n = 440 \cdot 2^{n/12} \quad (n = -5, -4, \ldots, 2)$$

pitch	frequency f_n [Hz]
C	261.6
D	293.6
E	329.6
F	349.2
G	392.0
A	440.0
B	493.8
c	523.2

Limits of Hearing



Source: R.Fay, *Hearing in Vertebrates: A Psychoacoustics Databook.* Hill-Fay Associates, 1988.

Vibrations in Musical Instruments

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- A freely vibrating body is described by the partial differential equation (*wave equation*)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad \text{(with initial \& boundary conditions)}$$

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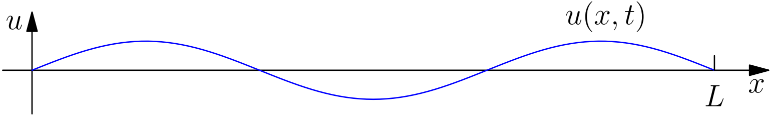
$$\nabla^2 = \text{Laplacian operator} = \begin{cases} \dfrac{\partial^2 u}{\partial x^2} & \text{on } \mathbb{R} \\ \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} & \text{on } \mathbb{R}^2 \end{cases}$$

Vibrations in Musical Instruments

Example: string fixed at both ends.



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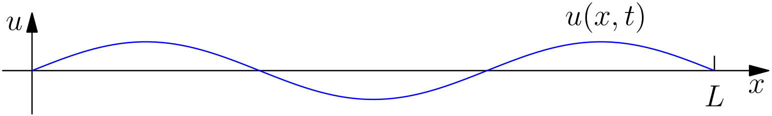

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \\ u(0,t) = u(L,t) = 0 & \text{(boundary conditions)} \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) & \text{(initial conditions)} \end{cases}$$

Wave equation: Separation of variables

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$$\begin{aligned} 0 &= \frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u \\ &= XT'' - c^2 T \nabla^2 X \end{aligned}$$

$$\implies \frac{1}{c^2}\frac{T''}{T} = \frac{\nabla^2 X}{X}$$

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$$\implies \underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{indep of } \mathbf{x}} = \underbrace{\frac{\nabla^2 X}{X}}_{\text{indep of } t} = \lambda \quad \text{(constant)}$$

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By linearity:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}ct + \phi_n) f_n(x)$$

where λ_n are eigenvalues, f_n are eigenfunctions of ∇^2.

Vibrations in Musical Instruments

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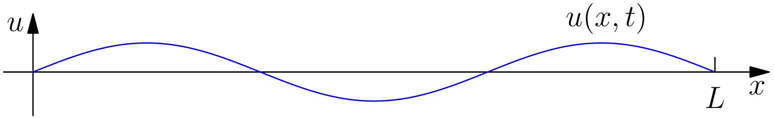
- Smallest λ_n gives the **fundamental tone**; other modes give **upper partials**

Example: string with fixed ends



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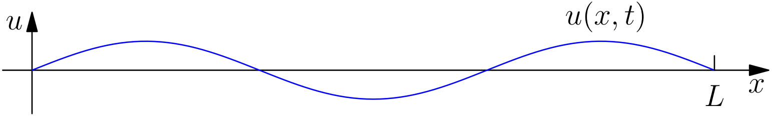

Eigenvalue problem $\nabla^2 f = \lambda f$ in 1-D becomes

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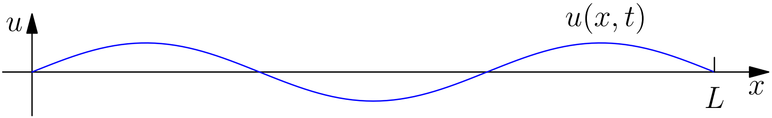

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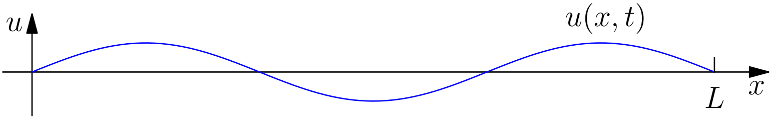

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Boundary condition $f(L) = 0$ gives

$$0 = A\sin(\sqrt{\lambda}L) \implies \sqrt{\lambda_n}L = n\pi \quad (n = 0, 1, 2, \ldots)$$

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Frequencies of vibrational modes are given by

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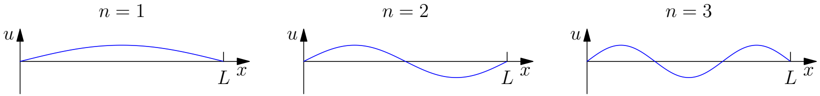

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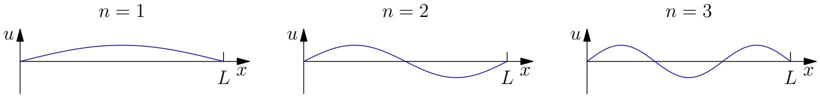

The string's motion is a superposition of these:

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t + \phi_n) f_n(x)$$

Example: string with fixed ends

- What we *hear* is a superposition of pure tones:

$$P(t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t + \phi_n)$$

at discrete frequencies

$$\omega_n = \frac{n\pi c}{L} = n\omega_1 \quad (n = 1, 2, 3, \ldots)$$

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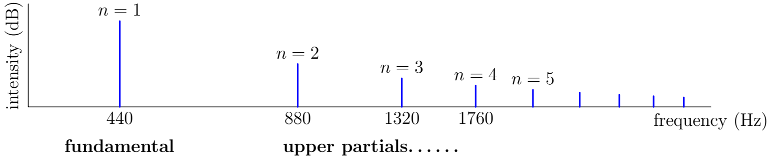
- Sound perception is independent of the phase ϕ_n.

Example: string with fixed ends

For a guitar string playing the note A (440 Hz) we hear:



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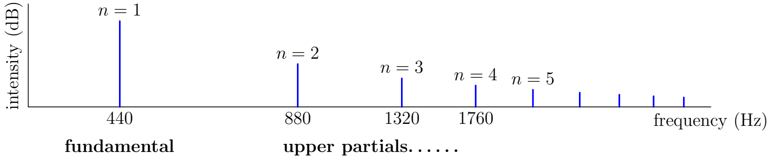
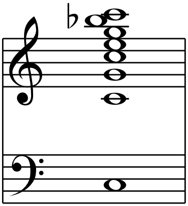


First 7 upper partials for low C:



Timbre

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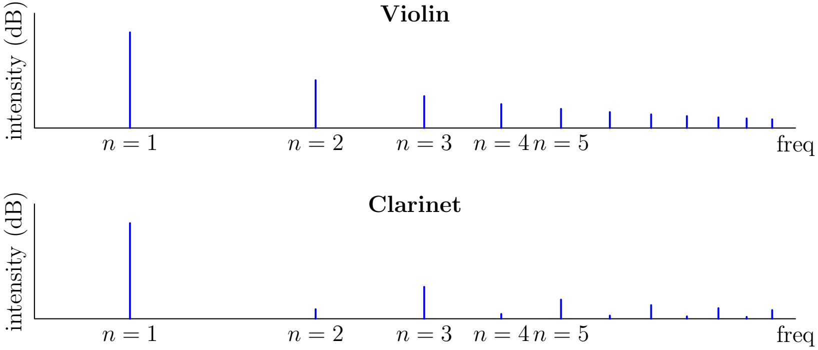
Timbres of different instruments are distinguished primarily by the frequencies and amplitudes of their spectra:



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- **Circular drums:**

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- **Vibrating bars:** (e.g. xylophone, marimba)

$$\begin{cases} \lambda_n = (2n+1)^4 & \text{(transverse virbations)} \\ \lambda_m = m^2 & \text{(longitudinal vibrations)} \end{cases}$$

Consonance & Dissonance

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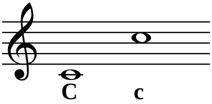
- Some intervals sound **consonant** ("good"?), others **dissonant** ("bad"?)
- Pythagoras: an interval is consonant if the frequencies are in a simple integer ratio:



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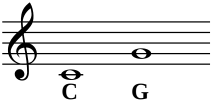
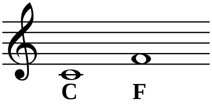


$f_1 : f_2 = 2 : 3$

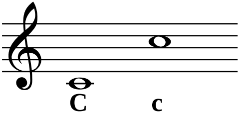


$f_1 : f_2 = 4 : 3$

- But why??

Consonance & Dissonance

A wrong explanation (Galileo and many others):



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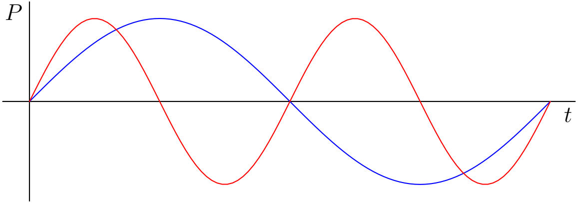
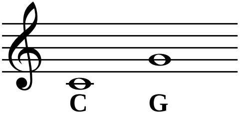


Consonance & Dissonance



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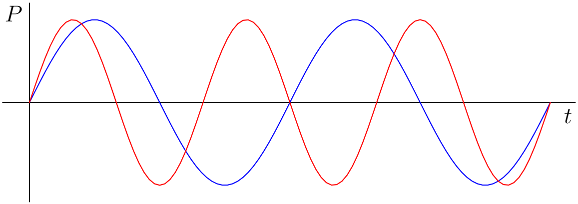

Consonance & Dissonance

"The pulses delivered by the two tones . . . shall be commensurable in number, so as not to keep the ear-drum in perpetual torment. . ."

Galileo Galilei
Dialogues Concerning Two New Sciences (1638)

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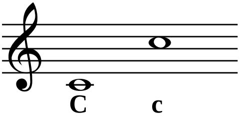
- However. . . for *pure tones* a mis-tuned interval isn't dissonant!
- The reality: dissonance comes from upper partials.

Consonance & Dissonance

A better explanation:



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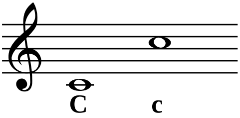

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Consider spectra of upper partials for these tones:

f_1: 220 Hz, 440 Hz, 660 Hz, 880 Hz, 1100 Hz, 1320 Hz, . . .
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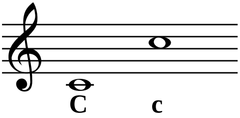

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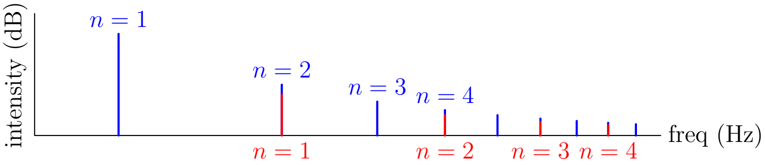

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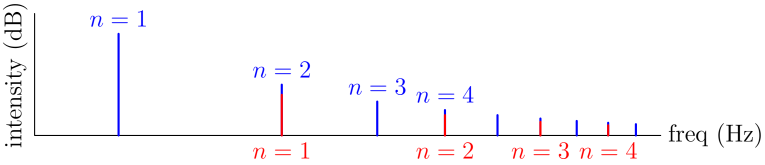

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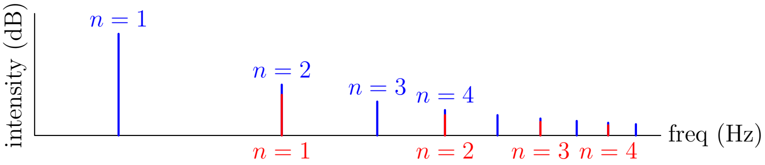
Consonance & Dissonance



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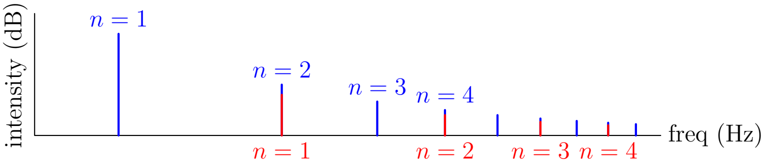
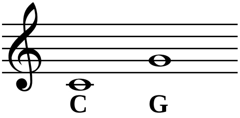


- Upper partials coincide and reinforce each other.
- The effect is one of altered timbre.
- Invididual tones are difficult to distinguish.

Consonance & Dissonance

Similarly for a perfect fifth:



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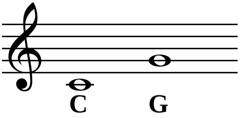

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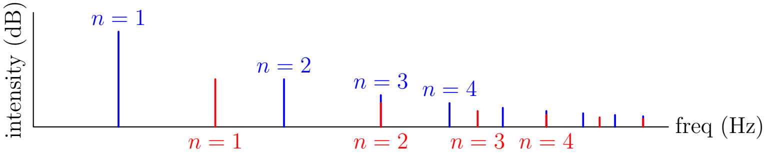
Again, some of the upper partials coincide:



at common multiples of the fundamentals.

Consonance & Dissonance

So generally. . .

- If the fundamentals are in the ratio

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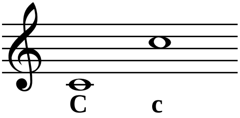
- Effect is more audible if the product mn is smaller.
- Simple integer ratios emerge as intervals with strongest mutual reinforement.
- But this doesn't really explain dissonance of other intervals.

Consonance & Dissonance

An even better explanation:



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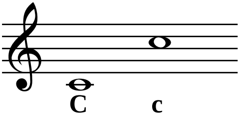

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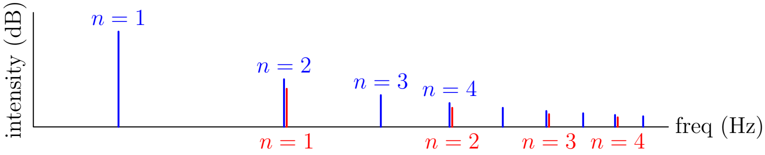
Consider spectra for a slightly mistuned octave:



Previously coincident partials now differ.

Consonance & Dissonance

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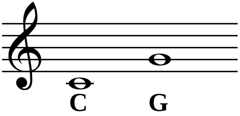

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Consonance & Dissonance

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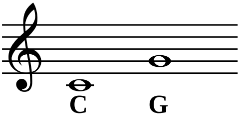

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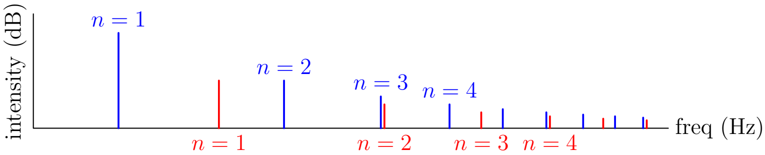

Beats

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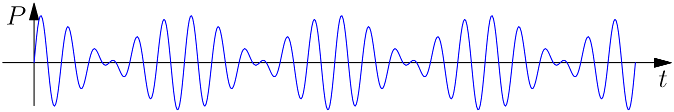

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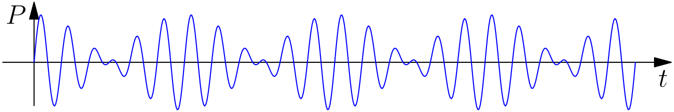

$$\sin(\omega_1 t) + \sin(\omega_2 t) = 2\underbrace{\cos\left(\frac{\omega_1 - \omega_2}{2} t\right)}_{\text{slow modulating term}} \sin\left(\frac{\omega_1 + \omega_2}{2} t\right)$$

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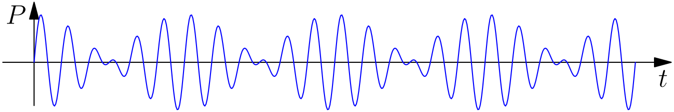

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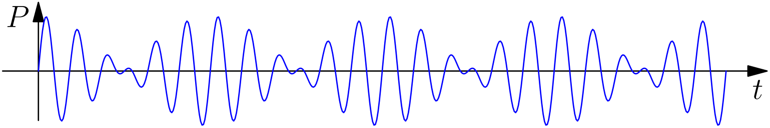

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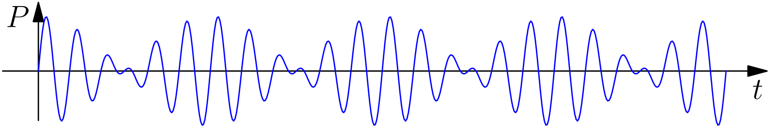

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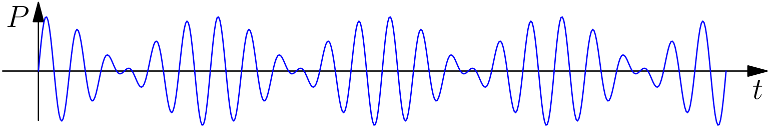

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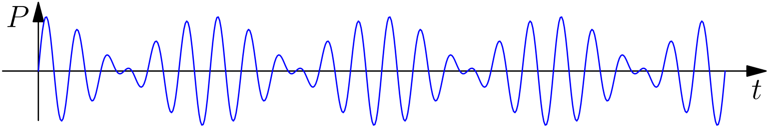

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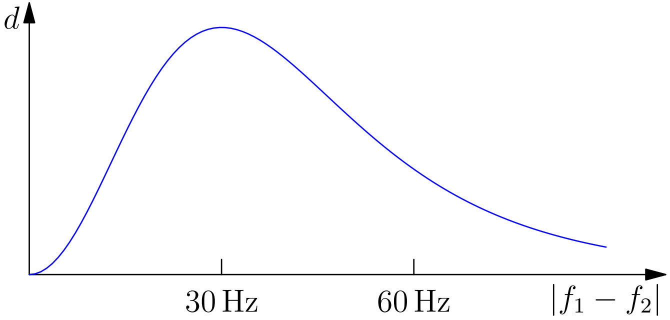

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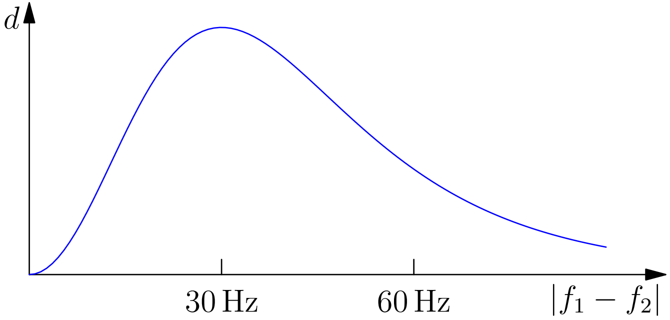

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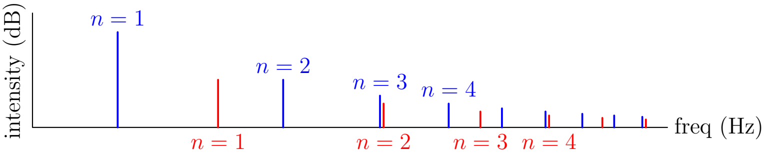
- We can model this with:

$$d(x) = \frac{(x/30)^2}{(1 + \frac{1}{3}(x/30)^2)^4}$$

Consonant Intervals

- When two notes are sounded, dissonance potentially arises from beating between every pair of partials.



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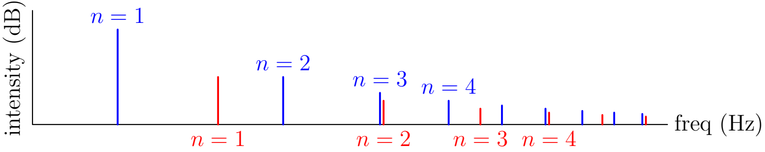

- Mistuned fifth ($f_1 : f_2 = 3 : 2$) with $f_1 = 220$ Hz and $f_2 = 335$ Hz:

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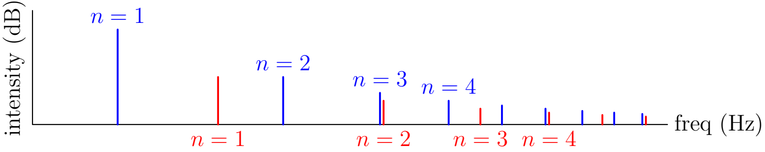

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- Near-concidence of upper partials causes beating:

$$670 - 660 = 10\,\text{Hz} \quad \text{and} \quad 1340 - 1320 = 20\,\text{Hz}.$$

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- Two notes with fundamental frequencies f_1 and f_2, hence two sequences of partials

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- Sum dissonances over all pairs of partials:

$$\text{total dissonance} = \sum_m \sum_n d(|mf_1 - nf_2|)$$
$$d(x) = \frac{(x/30)^2}{(1 + \frac{1}{3}(x/30)^2)^4}$$

Consonant Intervals

Fix f_1 and calculate dissonance as a function of f_2:

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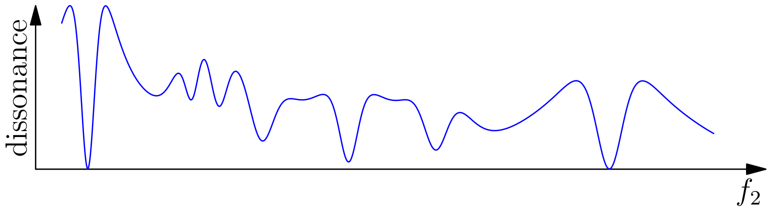
Summing over the first 7 partials:



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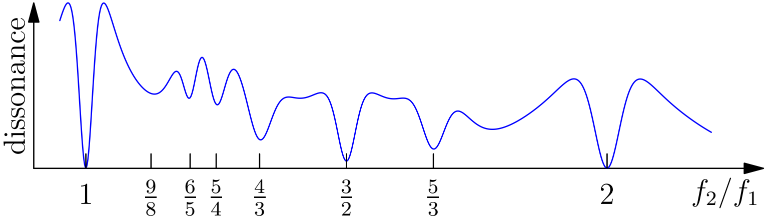

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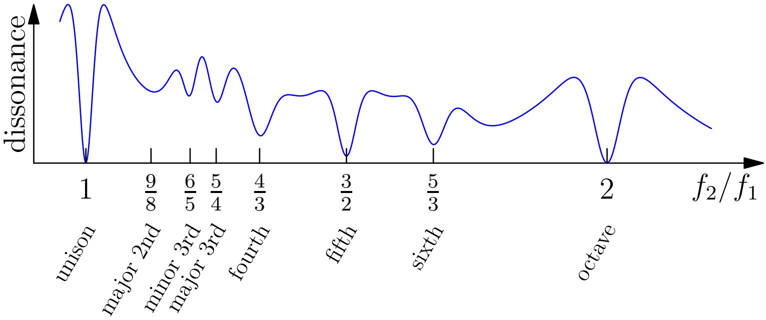

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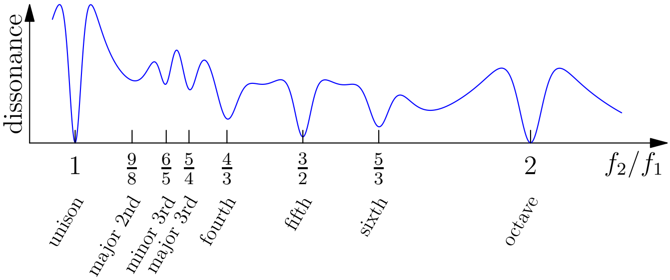
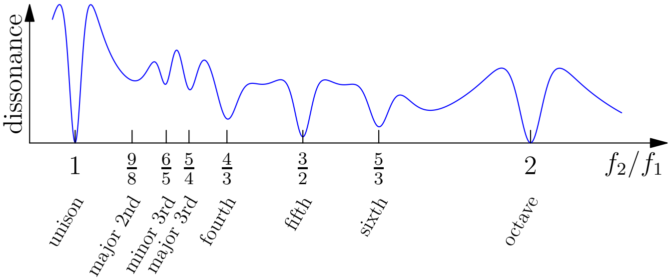


- Consonant intervals are local minima of dissonance.

Consonant Intervals



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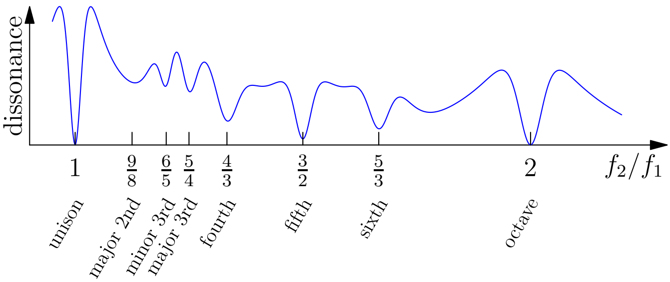

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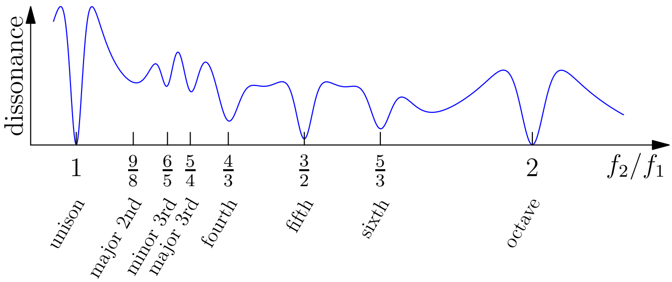

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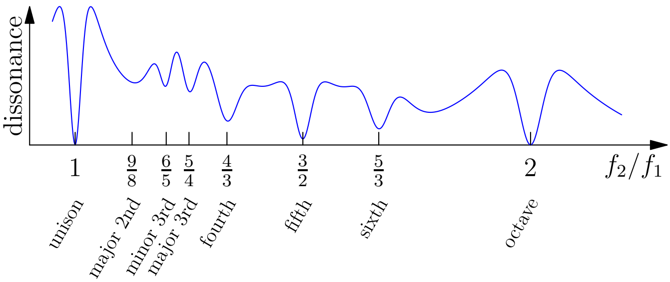

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- Dissonance curve is a consequence of the underlying timbre (spectrum) of the instrument.