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**Problem 1:** Consider approximating $f(x) = e^{-x^2}$ by interpolating from a table of values.

(a) Construct the Lagrange interpolating polynomial $P(x)$ of minimum degree such that $P(x)$ agrees with $f(x)$ at the four points given in the table below.

| $x$ | $f(x)$ |
|---|---|
| $x_0 = 0.0$ | 1.0000 |
| $x_1 = 0.5$ | 0.7788 |
| $x_2 = 1.0$ | 0.3679 |
| $x_3 = 1.5$ | 0.1054 |

$$L_1(x) = \frac{(x-.5)(x-1)(x-1.5)}{(0-.5)(0-1)(0-1.5)} = -1.333(x-.5)(x-1)(x-1.5)$$

$$L_2(x) = \frac{x(x-1)(x-1.5)}{.5(.5-1)(.5-1.5)} = 4x(x-1)(x-1.5)$$

$$L_3(x) = \frac{x(x-.5)(x-1.5)}{1(1-.5)(1-1.5)} = -4x(x-.5)(x-1.5)$$

$$L_4(x) = \frac{x(x-.5)(x-1)}{1.5(1.5-.5)(1.5-1)} = 1.333x(x-.5)(x-1)$$

$$P(x) = f(0)L_1(x) + f(.5)L_2(x) + f(1)L_3(x) + f(1.5)L_4(x)$$

$$= \boxed{\begin{aligned}&-1.333(x-.5)(x-1)(x-1.5) + 3.1152x(x-1)(x-1.5)\\ &-1.4716(x)(x-.5)(x-1.5) + 0.1405x(x-.5)(x-1)\end{aligned}}$$

(b) Approximate $f(0.7)$ by evaluating $P(0.7)$. What is the relative error in this approximation?

$$P(0.7) = -0.0640 + 0.5233 + 0.1648 - 0.00590$$

$$= \boxed{0.6183} \qquad \text{rel. err.} = \frac{.6183 - .6126}{.6126} = \boxed{0.0092}$$

(c) Use Neville's algorithm to calculate a quadratic interpolation of $f(x)$ to $x = 0.7$ using only the nodes $x_0$, $x_1$ and $x_2$.

$$\begin{array}{lllll}
x_0 = 0 & P_0 = 1.0000 & & & \\
x_1 = .5 & P_1 = .7788 & \longrightarrow P_{0,1} = .6903 & & \\
x_2 = 1 & P_2 = .3679 & \longrightarrow P_{1,2} = .6144 & \longrightarrow & \boxed{P_{0,1,2} = .6372}
\end{array}$$

$$P_{0,1}(.7) = \frac{(.7-0)P_1 - (.7-.5)P_0}{.5-0} = .6903$$

$$P_{1,2}(.7) = \frac{(.7-.5)P_2 - (.7-1)P_1}{1-.5} = .6144$$

$$P_{0,1,2}(.7) = \frac{(.7-0)P_{1,2} - (.7-1)P_{0,1}}{1-0} = .6372$$

/8 **Problem 2:** Consider the definite integral $I = \int_0^1 e^{-x^2}$.

(a) How many subintervals are needed to approximate $I$ with absolute error less than $10^{-4}$ using the trapezoid rule?

$f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$

$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$

$f'''(x) = 4xe^{-x^2} + 8xe^{-x^2} + 8x^3e^{-x^2}$

$= 4x(3-2x^2)e^{-x^2}$

$\therefore f''' = 0$ at $x=0, x = \pm\sqrt{\frac{3}{2}}$

So $|f''(x)|$ is maximum at

0 or 1: $|f''(0)| = 2$, $|f''(1)| = 2e^{-1}$

$\therefore |f''(x)| \leq 2$ for $x \in [0,1]$

Trapezoid error:

$\frac{(b-a)^3}{12n^2} f''(\mu) \leq \frac{1}{12n^2}(2) < 10^{-4}$

$\rightarrow n > \sqrt{10^4/6} = 40.8$

$\therefore \boxed{n > 41}$

(b) Use Simpson's Rule with $n = 4$ to approximate $I$.

$h = \frac{1}{4}$

$x_0 = 0$

$x_1 = .25$

$x_2 = .5$

$x_3 = .75$

$x_4 = 1$

Simpson's Rule:

$I \approx \frac{.25}{3}\left[f(0) + 4f(.25) + 2f(.5) + 4f(.75) + f(1)\right]$

$= \frac{.25}{3}\left[1 + 4(.9394) + 2(.7788) + 4(.5698) + .3679\right]$

$= \boxed{0.7469}$

(c) Given that $\left|\frac{d^4}{dx^4}(e^{-x^2})\right| < 12$ for all $x \in [0,1]$, find an upper bound on the absolute error in your answer to (b).

abs. err $= \frac{(b-a)^5}{180n^4} f^{(4)}(\mu) \leq \frac{1}{180(4)^4} \cdot 12 = \boxed{2.6 \times 10^{-4}}$

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**Problem 3:** Consider the following system of linear equations.

$$0.03x_1 + 58x_2 = 59$$
$$5.3x_1 - 6.1x_2 = 47$$

(whose solution is $x_1 \approx 10.03$, $x_2 \approx 1.012$).

(a) Use Gauss elimination (without row or column interchanges) and 2-digit chopping arithmetic to calculate an approximate solution.

$$m_{21} = \frac{5.3}{.03} \approx 1.7\times 10^2 \longrightarrow \begin{matrix} \\ R_2 - (1.7\times 10^2)R_1 \end{matrix} \begin{bmatrix} .03 & 58 & 59 \\ 0 & 9.8\times 10^3 & 9.9\times 10^3 \end{bmatrix}$$

back subs:

$$x_2 = \frac{9.9\times 10^3}{9.8\times 10^3} \approx 1.0$$

$$x_1 = \frac{59 - 58(1.0)}{.03} \approx 33$$

$$\longrightarrow \boxed{\begin{matrix} x_1 = 33 \\ x_2 = 1.0 \end{matrix}}$$

(b) What causes the large relative error in your answer to (a)?

small divisor (.03) magnifies errors

Problem 3 continued...

(c) Use one iteration of iterative refinement (again in 2-digit chopping arithmetic) to improve the approximation you calculated in (a).

$$\text{residual} \quad \underline{b} - A\underline{x} = \begin{bmatrix} 59 \\ 47 \end{bmatrix} - \begin{bmatrix} .03 & 58 \\ 5.3 & -6.1 \end{bmatrix}\begin{bmatrix} 33 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 1.0 \\ -110 \end{bmatrix}$$

$$A\underline{y} = \underline{r} \rightarrow \left[\begin{array}{cc|c} .03 & 58 & 1.0 \\ 5.3 & -6.1 & -110 \end{array}\right] \xrightarrow{R_2 - (1.7\times10^2)R_1} \left[\begin{array}{cc|c} .03 & 58 & 1.0 \\ 0 & -9.8\times10^3 & -2.8\times10^2 \end{array}\right]$$

back subs:

$$y_2 = \frac{2.8\times10^2}{9.8\times10^3} \approx 0.028$$

$$y_1 = \frac{1.0 - 58(.028)}{.03} \approx -20$$

$$\rightarrow \underline{x} \approx \begin{bmatrix} 33 \\ 1.0 \end{bmatrix} + \begin{bmatrix} -20 \\ .028 \end{bmatrix} = \begin{bmatrix} 13 \\ 1.0 \end{bmatrix} \longrightarrow \boxed{\begin{array}{l} x_1 = 13 \\ x_2 = 1.0 \end{array}}$$

(d) Solve the original system using complete pivoting (again with 2-digit chopping).

$$C_1 \leftrightarrow C_2 \quad \begin{array}{ccc} x_2 & x_1 & \\ \end{array}$$

$$\left[\begin{array}{cc|c} 58 & .03 & 59 \\ -6.1 & 5.3 & 47 \end{array}\right] \longrightarrow R_2 + 0.10R_1 \left[\begin{array}{cc|c} 58 & .03 & 59 \\ 0 & 5.3 & 52 \end{array}\right]$$

$$m_{21} = -\frac{6.1}{58} = -0.10$$

back subs:

$$x_1 = \frac{52}{5.3} \approx 9.8$$

$$x_2 = \frac{59 - .03(9.8)}{58} \approx \frac{58}{58} = 1.0 \longrightarrow \boxed{\begin{array}{l} x_2 = 1.0 \\ x_1 = 9.8 \end{array}}$$

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**Problem 4:** Use the Gauss-Seidel iteration method, starting with the initial approximation $\mathbf{x}^{(0)} = (0, 0, 0)$, to find an approximate solution of the linear system

$$
\begin{aligned}
3x_1 - x_2 + x_3 &= 1 \\
3x_1 + 6x_2 + 2x_3 &= 0 \\
3x_1 + 3x_2 + 7x_3 &= 4
\end{aligned}
$$

accurate to within an absolute error of 0.1 in every component.

$$
\begin{cases}
x_1 = (1 + x_2 - x_3)/3 \\
x_2 = (-3x_1 - 2x_3)/6 = -\frac{x_1}{2} - \frac{x_3}{3} \\
x_3 = (4 - 3x_1 - 3x_2)/7
\end{cases}
$$

$\underline{x}^{(0)} = (0, 0, 0)$

$\to \underline{x}^{(1)} = (.3333, -.1667, 0.5)$

$\to \underline{x}^{(2)} = (.1111, -.2222, .6190)$

max. change is .058...
(could stop here)

$\to \underline{x}^{(3)} = (.05293, -.2328, .6485)$

$\to \underline{x}^{(4)} = (.03957, -.2360, .6556)$

max. change is .0035

$\to \underline{x}^{(5)} = (.0361, -.2366, .6571)$

$\to \underline{x}^{(6)} = (.03543, -.2368, .6577)$ [accurate to 3 digits... more than required]

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**Problem 5:** Consider the matrix $A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$.

(a) Calculate an $LU$ decomposition of $A$.

$$m_{21} = 0 \quad R_2 - m_{21}R_1 \qquad m_{31} = \frac{1}{3} \quad R_3 - m_{31}R_1 \qquad \begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{bmatrix} \longrightarrow m_{32} = -3: \begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & -3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & 8 \end{bmatrix} \qquad \text{so } A = LU$$

(b) Use your answer to (a) to efficiently calculate the solution of $\begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

$$L\underbrace{U\underline{x}}_{\underline{y}} = \underline{b} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & -3 & 1 \end{bmatrix} \underline{y} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \longrightarrow \begin{aligned} y_1 &= 4 \\ y_2 &= 5 \\ y_3 &= 6 - \tfrac{1}{3}(4) + 3(5) = 19.67 \\ & (= 59/3) \end{aligned}$$

$$U\underline{x} = \underline{y} \rightarrow \begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & 8 \end{bmatrix} \underline{x} = \begin{bmatrix} 4 \\ 5 \\ 19.67 \end{bmatrix} \longrightarrow \begin{aligned} x_3 &= \frac{19.67}{8} = 2.458 \\ x_2 &= \frac{5 - 3(2.458)}{-1} = 2.375 \\ x_1 &= \frac{4 - 3(2.458)}{3} = -1.125 \end{aligned}$$

$$\therefore \boxed{\begin{aligned} x_1 &= -1.125 \\ x_2 &= 2.375 \\ x_3 &= 2.458 \end{aligned}} \begin{aligned} &(= -9/8) \\ &(= 19/8) \\ &(= 59/24) \end{aligned}$$