/2

**Problem 1:** Determine the decimal representation of the binary number $(11011.01)_2$.

$$2^4 + 2^3 + 2^1 + 2^0 + 2^{-2} = \boxed{27.25}$$

/3

**Problem 2:** A binary machine that carries 30 bits in the mantissa (i.e. fractional part) of each floating-point number is designed to round a given real number up or down correctly to get the nearest representable floating-point number. What simple upper bound can be given for the relative error in this rounding process?

- in the worst case, $x = 0.b_1b_2 \cdots b_{29}0100\cdots \times 2^n$
  so that $fl(x) = 0.b_1b_2\cdots b_{29}1 \times 2^n$
- then $\dfrac{|x - fl(x)|}{|x|} = 2^{-31}$

thus $\boxed{\text{rel. err.} \le 2^{-31} \approx 4.7\times 10^{-10}}$

/3

**Problem 3:** What is the order of convergence (in "big-O" notation) of the sequence $x_n = 1-\cos(1/n^3)$, as $n \to \infty$?

- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$

  so $1-\cos(x) = \dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \cdots$

- $1-\cos\left(\frac{1}{n^3}\right) = \dfrac{(1/n^3)^2}{2!} - \dfrac{(1/n^3)^4}{4!} + \cdots = \overbrace{\dfrac{n^{-6}}{2!} - \dfrac{n^{-12}}{4!} + \cdots}^{< \frac{n^{-6}}{2!} \text{ by alternating series}}$

  $= \boxed{O(n^{-6})}$

/8

**Problem 4:** Consider the function $f(x) = x - \sin x$.

(a) Evaluate $f(0.2)$ using 3-digit (decimal) rounding arithmetic.

$$0.2 - 0.199 = 0.001 = \boxed{1.00 \times 10^{-3}}$$

(b) What are the relative error and number of significant figures in your result from part (a)? What causes the large relative error in this calculation?

- $f(0.2) \doteq 1.33067 \times 10^{-3}$

  so rel. err. is $\dfrac{1.33067 - 1.00}{1.33067} \approx \boxed{0.25}$

- $0.25 < 5 \times 10^{-1}$ so $\boxed{\text{one significant figure}}$
- large error due to subtracting nearly-equal numbers

(c) One way to compute $f(0.2)$ while retaining more significant figures is to use a Taylor polynomial. Find the 5th-order Taylor polynomial $P_5(x)$ for $f(x)$ based at $x_0 = 0$, and evaluate $P_5(0.2)$ using 3-digit rounding arithmetic.

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$ so $x - \sin x = \dfrac{x^3}{3!} - \dfrac{x^5}{5!} + \cdots$

$$\therefore \boxed{P_5(x) = \frac{x^3}{6} - \frac{x^5}{120}}$$

$$P_5(0.2) = 1.33 \times 10^{-3} - 2.67 \times 10^{-6} = \boxed{1.33 \times 10^{-3}}$$

(d) What are the relative error and number of significant figures in using your result from part (c) to approximate $f(0.2)$?

rel. err. is $\dfrac{1.33067 - 1.33}{1.33067} \approx \boxed{0.000503}$

rel. err. $< 5 \times 10^{-3}$ so $\boxed{\text{3 significant digits}}$

/6

**Problem 5:** Consider the problem of finding the $x$-coordinate of the intersection point of the graphs of $y = 3x$ and $y = e^x$.

(a) The Intermediate Value Theorem guarantees the existence of a solution in the interval $[1, 2]$. Starting with this interval, what is the minimum number of iterations of the bisection method needed to obtain a solution correct to 6 decimal digits?

- need to solve $\underbrace{e^x - 3x}_{f(x).} = 0$
- bisection guarantees $|p_n - x| \leq \frac{b-a}{2^n}$
- so require $\frac{2-1}{2^n} < 10^{-6}$

$$\rightarrow \quad 2^n > 2\times 10^6$$

$2^n = 2\times10^6$ gives $n = \log_2(2\times10^6) = 20.93$

$\therefore$ need $\boxed{n = 21}$

(b) Find the solution correct to 6 decimal digits using Newton's method.

- Newton's method gives fixed-pt. iteration $x_{n+1} = g(x_n)$ with $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{e^x - 3x}{e^x - 3}$
- start with $x_0 = 1.5 \ldots$

| $n$ | $x_n$ | $\|x_n - x_{n-1}\|$ |
|---|---|---|
| 0 | 1.5 | |
| 1 | 1.5123581 | $1.2\times10^{-3}$ |
| 2 | 1.5121346 | $2.2\times10^{-4}$ |
| 3 | 1.5121346 | $7.4\times10^{-8}$ |

$\boxed{x \approx 1.51213}$

/6

**Problem 6:** We saw in class that using Newton's method to approximate $\sqrt{2}$ yields the fixed-point iteration scheme $x_{n+1} = g(x_n)$ with $g(x) = \frac{x}{2} + \frac{1}{x}$.

(a) Prove that for any $x_0 \in [1,2]$ the sequence $\{x_n\}$ converges.

- $g'(x) = \frac{1}{2} - \frac{1}{x^2}$ which is monotonically decreasing on $[1,2]$ (no critical points)

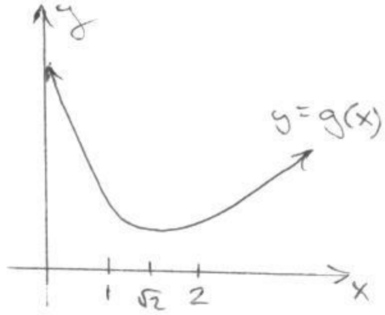
$g'(2) = \frac{1}{4}$

$g'(1) = -\frac{1}{2}$

so $|g'(x)| \le \frac{1}{2} < 1$ for all $x \in [1,2]$

- $\min_{x\in[0,1]} g(x) = g(\sqrt{2}) = \sqrt{2} > 1$

$\max_{x\in[0,1]} g(x) = g(1) = g(2) = 1.5 < 2$




- so $g([1,2]) \subset [1,2]$.
- $\therefore$ By Fixed-Point Thm, $\{x_n\}$ converges to unique fixed pt.

(b) Find an upper bound on $|g'(x)|$ and use this to estimate the number of iterations needed to evaluate $\sqrt{2}$ correct to 100 decimal digits.

- we have $|x_n - x| \le \frac{k^n}{1-k} \underbrace{|x_1 - x_0|}_{<1}$ with $k = \frac{1}{2}$

so require $\frac{\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} < 10^{-100}$

$\rightarrow \left(\frac{1}{2}\right)^n < \frac{1}{2} \times 10^{-100}$

setting $\left(\frac{1}{2}\right)^n = \frac{1}{2} \times 10^{-100}$ gives $n = 333.2$

so need $\boxed{n \geq 334}$

/4

**Problem 7:** It is easily verified that one root of the polynomial $P(x) = x^4 + 5x^3 - 9x^2 - 85x - 136$ is $x \approx 4.1231$. Use deflation (by synthetic division) to find a polynomial $Q(x)$ of degree 3 whose roots are the other 3 roots of $P$.

synthetic division:

$$
\begin{array}{r|rrrrr}
4.1231 & 1 & 5 & -9 & -85 & -136 \\
 & & 4.1231 & 37.6155 & 117.9844 & 135.997 \\
\hline
 & \underbrace{1 \quad 9.1231 \quad 28.6155 - 32.9844} & & & & \underbrace{2\times10^{-3}}
\end{array}
$$

$\approx 0$ since 4.1231 is a root (i.e. $P(4.1231) = 0$

coefficients of

$$Q(x) = \frac{P(x)}{x - 4.1231}$$

$$= \boxed{x^3 + 9.1231x^2 + 28.6155x + 32.9844}$$

/3 **Problem 8:** Show that for any real number $k > 1$ the sequence $x_n = \frac{1}{k^n}$ converges linearly to 0 as $n \to \infty$.

$$\frac{|x_{n+1}|}{|x_n|} = \frac{k^{-(n+1)}}{k^{-n}} = \frac{k^n}{k^{n+1}} = \frac{1}{k} \to \frac{1}{k} \text{ as } n \to \infty.$$

Since $0 < \frac{1}{k} < 1$, the sequence converges linearly.

/3 **Problem 9:** Construct a sequence that converges to 0 with order of convergence 3.

- require $\frac{|x_{n+1}|}{|x_n|^3} \to \lambda$ as $n \to \infty$, for some $\lambda$.

- for simplicity take $\lambda = 1$ and all $x_n > 0$,

  then for large $n$, $x_{n+1} \approx x_n^3$

  so $x_1 = x_0^3$

  $x_2 = x_1^3 = (x_0)^3 = x_0^9$

  $x_3 = x_2^3 = (x_0^9)^3 = x_0^{27}$ ... inductively $x_n = x_0^{3^n}$

- to get convergence need $0 < x_0 < 1$ so take e.g. $x_0 = \frac{1}{10}$:

$\to$ $\boxed{x_n = 10^{-3^n}}$