Name: ______________________________ Student #: ________________

THOMPSON RIVERS UNIVERSITY

MATH 365
Numerical Analysis

Instructor: Richard Taylor

FINAL EXAM

9 December 2006 14:00–17:00

PROBLEM	GRADE	OUT OF
1		3
2		6
3		3
4		3
5		3
6		11
7		5
8		3
9		6
10		10

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 12 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.

/3 **Problem 1:** A naïve aproach to evaluating the polynomial $P(x) = x^{127} - 5x^{37} + 10x^{17} - 3x^7$ requires 187 multiplications. What formula could you use to evaluate $P(x)$ more efficiently, and what is the minimum number of multiplications required?

/6 **Problem 2:** Suppose you need to evaluate $f(x) = \sqrt{x^4 + 4} - 2$ for x near 0.

(a) Show that a direct calculation of $f(0.5)$ using the definition of $f(x)$ with 3-digit rounding arithmetic can lead to large relative errors. Why is this?

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(b) Derive an alternative formula for $f(x)$ that has better round-off error properties. Illustrate by using your formula to calculate $f(0.5)$ and the corresponding relative error.

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/3 **Problem 3:** Show by an example that in fixed-point arithmetic, $a + (b + c)$ may differ from $(a + b) + c$.

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Problem 4: If a Taylor series (based at $x = 0$) for $\sin x$ is used to evaluate $\sin(3.0)$, how many terms must be used to ensure a relative error less than 10^{-8}?

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Problem 5: (a) What is the largest integer p such that $\cos(h^2) - 1 + \frac{1}{2}h^4 = O(h^p)$ as $h \to 0$?

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Problem 6: Let λ be the smallest positive real solution of $\tan\lambda = -\lambda$, with λ in radians. (This and similar equations arise often in eigenvalue problems in physics and engineering.)

(a) Use the bisection method to find λ accurate to two significant digits.

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(b) In part (a), how many iterations of the bisection method would be required to guarantee that the relative error in the solution is less than 10^{-6}?

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(c) Using Newton's method, together with your solution from part (a) as an initial approximation, find λ with relative error less then 10^{-6}.

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(d) Recall that Newton's method generally gives quadratic convergence. Use this fact to estimate the number of iterations that would be required in part (c) to give an absolute error less than 10^{-32}.

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Problem 7: Suppose you were to implement an algorithm for solving $f(x) = 0$, in which one iteration consisted of *two* iterations of Newton's method.

(a) What would be the order of convergence of such a method (i.e., linear, or quadratic, or...)?

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(b) With reference to your answer to part (a), answer the following: Why is the order of convergence not the only important criterion in assessing the speed of convergence of an algorithm?

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Problem 8: Construct a sequence $\{x_n\}$, convergent to 0, whose rate of convergence is *slower* than linear.

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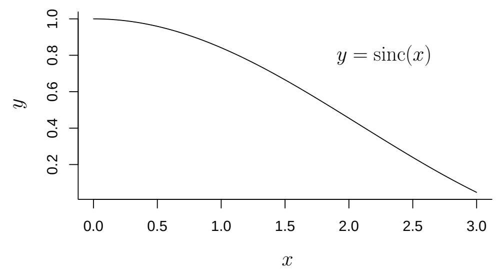
Problem 9: A possible improvement of Newton's method (to speed up convergence when the initial approximation x_0 is far from the solution x^* of $f(x) = 0$) proposes to iterate

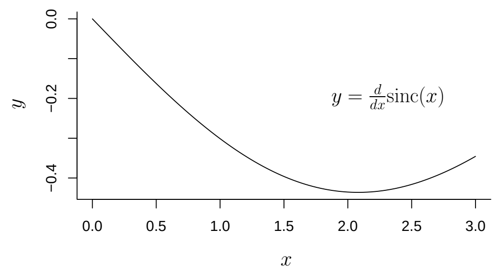
$$x_{n+1} = x_n - \omega \frac{f(x_n)}{f'(x_n)}$$

where $\omega > 0$ is a given number. This gives a fixed-point method $x_{n+1} = g(x_n)$ with $g(x) = x - \omega f(x)/f'(x)$.

Assume that $f'(x)$ is continuous and that $f'(x^*) \neq 0$. For what values of ω will this method converge to the solution x^* provided x_0 is sufficiently close to x^*?

/10 **Problem 10:** Let $\operatorname{sinc}(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{\sin t}{t} & \text{if } t \neq 0. \end{cases}$ The function $f(x) = \displaystyle\int_0^x \operatorname{sinc}(t)\,dt$ plays an important role in signal processing, optics, and other applications. Graphs of $\operatorname{sinc}(x)$ and $\frac{d}{dx}\operatorname{sinc}(x)$ are shown below.





(a) If $f(x)$ is to be evaluated using the composite trapezoid rule with n subintervals for the definite integral, what is the minimum n such that the absolute error will be less than 10^{-6} for any x in the interval $[0, 2]$?

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(b) Use the composite Simpson's rule with $n = 6$ to evaluate $f(1.2)$.

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(c) An alternative approach is to interpolate $f(x)$ from a table of previously computed values. The following table provides values known to be accurate to the number of digits shown. Use Neville's algorithm to perform cubic interpolation of $f(1.2)$.

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x	$f(x)$
$x_0 = 1.0$	0.9461
$x_1 = 1.5$	1.3247
$x_2 = 1.0$	1.6054
$x_3 = 1.5$	1.7785

/10 **Problem 11:** Consider the linear system

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 &= \tfrac{3}{2} \\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 &= \tfrac{5}{6} \end{aligned}$$

and answer the following questions using 2-digit chopping arithmetic.

(a) The vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2) = (1.4, 0.35)$ is proposed as an approximate solution. Calculate the corresponding residual vector.

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(b) Find the condition number of the coeffient matrix for this system, in the $\|\cdot\|_\infty$ norm.

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(c) The exact solution is $\mathbf{x} = (1, 1)$. What accounts for the large error in the approximate solution $\tilde{x}$ despite the small residual you found in part (a)?

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(d) Calculate an improved approximate solution by iterative refinement.

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/7 **Problem 12:** Consider the Hilbert matrix of order 3:

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

(a) Using 3-digit rounding arithmetic, calculate an LU decomposition of A.

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/7 **Problem 13:** Consider the linear system

$$\begin{aligned} 0.12x_1 + 3.1x_2 &= 1 \\ x_1 + 0.53x_2 + 5.4x_3 &= 2 \\ 0.65x_2 - 2.5x_3 &= 3 \end{aligned}$$

(a) If this system is to be solved using fixed-point arithmetic, why would naive Gauss elimination be an inferior choice of method?

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(b) Solve this system using Gauss elimination with complete pivoting and 2-digit rounding arithmetic.

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Problem 14: The system of equations

$$\sin(x + y) = e^{x-y}$$
$$\cos(x + 6) = x^2 y^2$$

has a solution near $x \approx 0.4$, $y \approx 2.7$. Use one iteration of Newton's method to obtain a more accurate approximate solution.

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Problem 15: Yet another approach to evaluating the function $f(x)$ in Problem 10 is based on realizing that f is satisfies the initial value problem

$$\frac{df}{dx} = \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{\sin x}{x} & \text{if } x \neq 0 \end{cases}$$
$$f(0) = 0.$$

(a) Use the Modified Euler Method with time step $h = 0.2$ to estimate $f(1.2)$. Compare with your answers from Problem 10.

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(b) Note that the right-hand side of the differential equation above does not depend explicitly on f. Consequently, the Modified Euler method gives the same value for $f(1.2)$ as one of the standard methods for evaluating definite integrals. Which method? Using how many subintervals?

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