THOMPSON RIVERS UNIVERSITY

MATH 211
Calculus III

Instructor: Richard Taylor

MIDTERM EXAM #1
SOLUTIONS

10 October 2007   11:30–12:30

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 4 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		10
2		3
3		12
4		8
TOTAL:		33

/10

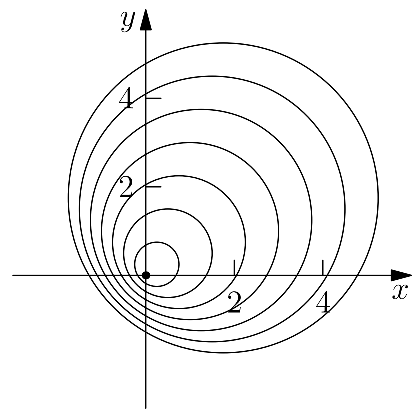
Problem 1: Consider the surface in $\mathbb{R}^3$ given by the graph of the equation $4z^2 = (x-z)^2 + (y-z)^2$ for $z \geq 0$.

(a) Sketch some level curves of this surface. Describe the surface.

$z = c \implies (2c)^2 = (x-c)^2 + (y-c)^2$

(a circle of radius $2c$ centered at (c, c))

- radius grows linearly with z: graph of f is a tilted cone with its vertex at the origin



(b) Find an equation for the plane tangent to the surface at the point $(13, 11, 5)$.

- the surface is a level surface of $g(x,y,z) = (x-z)^2 + (y-z)^2 - 4z^2$
- the normal $\mathbf{n}$ is in the direction of ∇g:

$$\nabla g = \big(2(x-z), 2(y-z), -2(x-z) - 2(y-z) - 8z\big)$$
$$\implies \mathbf{n} = \nabla g(13, 11, 5) = (16, 12, -68)$$

- the equation of the plane is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$:

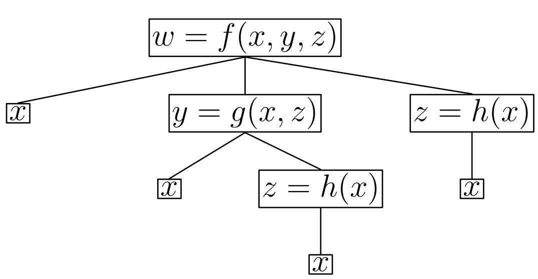
$$\implies (16, 12, -68) \cdot (x - 13, y - 11, z - 5) = 0$$
$$\implies \boxed{16x + 12y - 68z = 0 \quad \text{or} \quad 4x + 3y - 17z = 0}$$

(c) Use implicit differentiation to find an expression for $\dfrac{\partial z}{\partial x}$.

$$8z\frac{\partial z}{\partial x} = 2(x-z)(1 - \frac{\partial z}{\partial x}) - 2(y-z)\frac{\partial z}{\partial x}$$
$$\implies \big[8z + 2(x-z) + 2(y-z)\big]\frac{\partial z}{\partial x} = 2(x-z)$$
$$\implies \frac{\partial z}{\partial x} = \frac{2(x-z)}{8z + 2(x-z) + 2(y-z)} = \boxed{\frac{x-z}{x+y+2z}}$$

/3

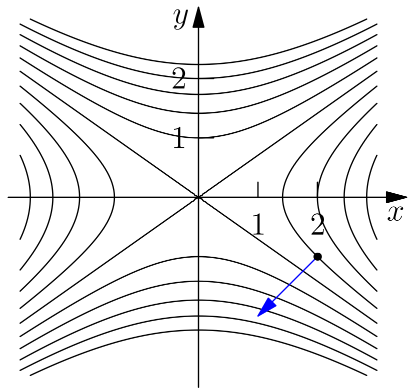
Problem 2: Write an appropriate version of the chain rule for $\dfrac{dw}{dx}$ if $w = f(x,y,z)$ where $y = g(x,z)$ and $z = h(x)$.



$$\frac{dw}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{dh}{dx}\right) + \frac{\partial f}{\partial z}\frac{dh}{dx}$$

/12

Problem 3: The temperature T (in °C) at points in the xy-plane is given by $T(x,y) = x^2 - 2y^2$ (with x and y in cm). Some isotherms (i.e. curves of constant T) are shown below. An ant is at the point $(2,-1)$.



(a) In what direction should the ant move if it wishes to cool off as quickly as possible? Indicate the ant's direction of motion on the graph above.

$$\nabla T = (2x, -4y) \implies \nabla T(2,-1) = (4,4) \quad \text{(direction of fastest increase)}$$

$$\therefore \text{ direction of fastest } \textit{decrease} \text{ is } -\nabla T = \boxed{(-4,-4)}$$

(b) If the ant moves in that direction with speed $v = 3\,\text{cm/s}$, at what rate does it experience the decrease in temperature?

solution 1:

$$|\nabla T| = \sqrt{(-4)^2 + (-4)^2} = \sqrt{32}\ ^\circ\text{C/cm}$$

$$\frac{dT}{dt} = |\nabla T| v = (\sqrt{32}\ ^\circ\text{C/cm})(3\text{ cm/s}) = \boxed{3\sqrt{32}\ ^\circ\text{C/s}}$$

solution 2:

$$\mathbf{v} = 3\mathbf{u} = 3 \cdot \frac{(-4,-4)}{\sqrt{4^2+(-4)^2}} = \left(-\frac{12}{\sqrt{32}}, -\frac{12}{\sqrt{32}}\right) \text{ cm/s}$$

$$\frac{dT}{dt} = \nabla T \cdot \mathbf{v} = (4,4) \cdot \left(-\frac{12}{\sqrt{32}}, -\frac{12}{\sqrt{32}}\right) = \boxed{-3\sqrt{32}\ ^\circ\text{C/s} = -12\sqrt{2}\ ^\circ\text{C/s} \approx -16.97\ ^\circ\text{C/s}}$$

(c) At what rate would the ant experience the change in temperature if it moved with velocity vector $\mathbf{v} = (2,-1)$ cm/s?

$$\frac{dT}{dt} = \nabla T \cdot \mathbf{v} = (4,4) \cdot (2,-1) = \boxed{4\ ^\circ\text{C/s (increasing)}}$$

(d) Find the linearization $L(x,y)$ of the temperature at the point $(2,-1)$.

$$T(2,-1) = 2 \implies \boxed{L(x,y) = 4(x-2) + 4(y+1) + 2 = 4x + 4y - 2}$$

/8

Problem 4: Consider the function

$$f(x,y) = x^4 + y^4 - 4xy + 2.$$

(a) Find the critical points of f.

$$f_x = 4x^3 - 4y = 0 \implies y = x^3$$
$$f_y = 4y^3 - 4x = 0 \implies 4(x^3)^3 - 4x = 0 \implies 4x(x^8 - 1) = 0$$
$$\implies x = 0 \text{ or } \pm 1$$

Therefore the critical points are:

$$\boxed{(0,0), \quad (-1,-1) \quad \text{and} \quad (1,1)}$$

(b) Classify each critical point of f as either a local minimim, local maximum, or saddle point.

$$f_{xx} = 12x^2 \qquad f_{yy} = 12y^2 \qquad f_{xy} = -4$$

At $(0,0)$:

$$D = f_{xx}f_{yy} - [f_{xy}]^2 = (0)(0) - (-4)^2 = -16 < 0 \implies \boxed{(0,0) \text{ is a } \textit{saddle point}}$$

At $(-1,-1)$:

$$D = (12)(12) - (-4)^2 > 0 \text{ and } f_{xx} = 12 > 0 \implies \boxed{(-1,-1) \text{ is a } \textit{local minimum}}$$

At $(1,1)$:

$$D = (12)(12) - (-4)^2 > 0 \text{ and } f_{xx} = 12 > 0 \implies \boxed{(1,1) \text{ is a } \textit{local minimum}}$$