Thompson Rivers University

MATH 211
Calculus III

Instructor: Richard Taylor

FINAL EXAM
SOLUTIONS

11 December 2009 09:00–12:00

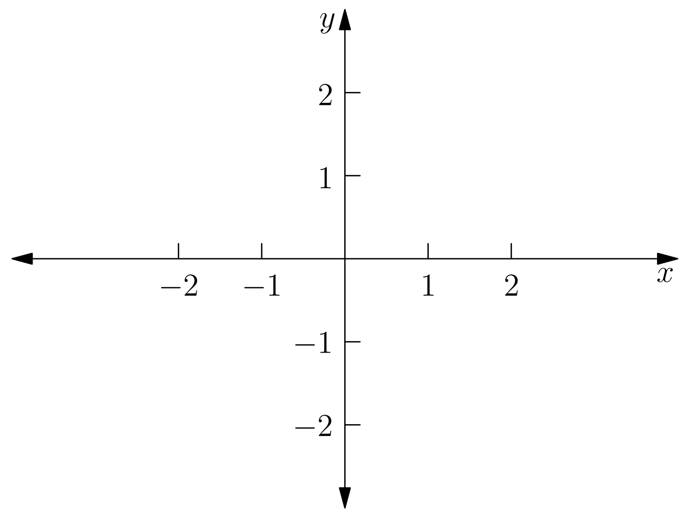
Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 11 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		5
2		4
3		4
4		10
5		7
6		5
7		5
8		8
9		6
10		6
11		8
12		8
13		8
14		6
TOTAL:		90

/5 **Problem 1:** Consider the surface defined by the graph of $f(x, y) = 4 - \sqrt{x^2 + 4y^2}$.

(a) Sketch some level curves of this surface.



(b) At the point $(1, 2)$ on the graph above, indicate the direction in which f decreases most rapidly .

/4 **Problem 2:** Calculate $\dfrac{\partial^3 f}{\partial x \partial y \partial z}\left(0, \dfrac{\pi}{6}, 0\right)$ for the function $f(x, y, z) = \cos(4x + 3y + 2z)$.

Ans: 24

/4 **Problem 3:** Write chain rules for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $z = f(r, s, x)$ where $r = g(x, y)$ and $s = h(x, y)$.

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial x}$$

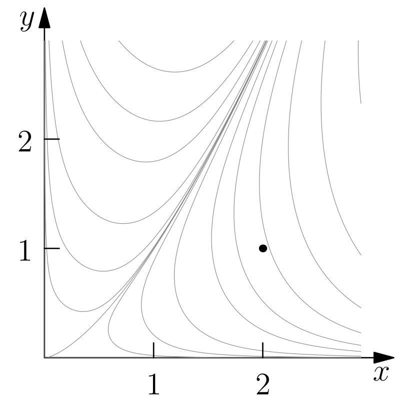
$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial s}\frac{\partial h}{\partial y}$$

/10 **Problem 4:** A sheet of metal is heated so that the temperature (in °C) at any point on its surface is given by the function

$$T(x, y) = x^4 y - xy^3$$

where x and y are in cm. Some level curves of T are shown.

An ant is located at the point $(2, 1)$.



(a) If the ant is to increase its temperature most rapidly, along what direction should it move? Indicate this direction on the graph above.

/3

Ans: $\nabla T(2, 1) = (31, 10)$

(b) If the ant moves along the curve $\mathbf{r}(t) = (2t + 2, 3t^4 - 5t + 1)$ (with t in seconds), what rate of change of temperature does it experience at the point $(2, 1)$?

/3

Ans: $dT/dt = 12\,°\text{C/s}$

(c) At the point $(2, 1)$, what is the rate of change of T in the direction of the vector $(-3, 1)$?

/2

Ans: $D_{\mathbf{u}}T = -83/\sqrt{10}\,°\text{C/cm}$

(d) At the point $(2, 1)$, what is the rate of change of T in the direction of fastest increase?

/2

Ans: $|\nabla T| = \sqrt{1061} \approx 32.6\,°\text{C/cm}$

/7 **Problem 5:** Consider the surface defined by the graph of

$$x^2 + y^2 + 3z = 10.$$

(a) Find an equation for the tangent plane at the point $(1, 0, 3)$.

/3

Ans: $2x + 3z = 11$

(b) Find an equation for the normal line at the point $(1, 0, 3)$.

/2

Ans: $(x, y, z) = (1, 0, 3) + t(2, 0, 3), \quad t \in \mathbb{R}$

(c) Find the point where the normal line intersects the yz-plane.

/2

Ans: $(0, 0, 3/2)$

/5 **Problem 6:** Find the critical points of the function

$$f(x, y) = 2xy^2 + 3xy + x^2y^3.$$

Ans: $(0, 0)$, $(0, -3/2)$, $(-4/3, 3/2)$

/5 **Problem 7:** Consider the function

$$f(x, y) = x^4 + y^4 - x^2 - y^2 + 1.$$

/2 (a) Show that $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is a critical point for f.

/3 (b) Classify the critical point $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ as a local minimum, local maximum, or saddle point of f.

Ans: local min

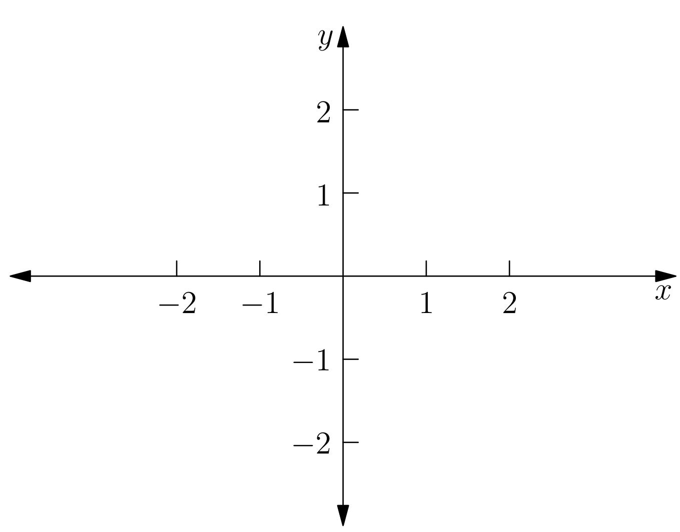
/8 **Problem 8:** (a) Find the maximum and minimum values of

$$f(x, y) = x^2 + y \quad \text{subject to} \quad x^2 + y^2 = 4.$$

Ans: $f(0, -2) = -2$ is the min; $f(\pm\sqrt{15}/2, 1/2) = 17/4$ is the max

(b) Sketch the curve $x^2 + y^2 = 4$ and indicate the points on this curve where the maximum and minimum values of f occur. Sketch the level curves of f that pass through these points. Indicate the direction of ∇f at the these points.

/3



/6 **Problem 9:** Let $I = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} f(x,y)\,dy\,dx.$

(a) Sketch and shade the regions of integration for the double integrals.

/3

(b) Express I as a *single* double integral by reversing the order of integration.

/3

$$I = \int_{-1}^{2} \int_{y^2}^{y+2} f(x,y)\,dx\,dy$$

/6 **Problem 10:** Find the volume of the region enclosed above the cone

$$z = \sqrt{x^2 + y^2}$$

and below the hemisphere

$$z = \sqrt{8 - x^2 - y^2}.$$

Ans: $V = 32\pi(\sqrt{2} - 1)/3$

/8

Problem 11: Find the absolute maximum and minimum values of the function

$$f(x, y) = x^3 - y^2 - 48x + 24$$

on the circular region

$$D = \{(x, y) : (x - 4)^2 + y^2 \leq 16\}.$$

Ans: $f(8, 0) = 128$ is the max, $f(4, \pm 4) = -120$ is the min

/8 **Problem 12:** The position of a particle at time $t \geq 0$ is given by the vector function

$$\mathbf{r}(t) = (3t, 3t^2, 2t^3 + 4).$$

(a) Find the velocity of the particle at $t = 2$.

/2

Ans: $(3, 12, 24)$

(b) Find the angle (to the nearest degree) between the velocity and acceleration vectors at $t = 2$.

/3

Ans: $14°$

(c) Find the arc length of the path the particle follows from $t = 1$ to $t = 2$.

/3

Ans: 17

/8 **Problem 13:** Consider the curve defined by the vector function

$$\mathbf{r}(t) = (\tfrac{1}{3}t^3, \tfrac{1}{2}t^2, 1).$$

/2 (a) Find an equation for the tangent line at the point $\left(\frac{8}{3}, 2, 1\right)$.

Ans: $(x, y, z) = (8/3, 2, 1) + t(4, 2, 0), \quad t \in \mathbb{R}$

/3 (b) Find an equation for the line that extends along the principal normal at the point $\left(\frac{1}{3}, \frac{1}{2}, 1\right)$.

Ans: $(x, y, z) = (1/3, 1/2, 1) + t(1, 2, 0), \quad t \in \mathbb{R}$

/3 (c) Find the curvature $\kappa(t)$.

Ans: $\kappa(t) = \dfrac{t^2}{(t^4 + t^2)^{3/2}}$

/6

Problem 14: Evaluate the following double integrals.

/3 (a) $\iint_R y\sin(xy)\,dA$ where $R = \{(x,y) : 0 \le x \le \frac{1}{2},\ 0 \le y \le \pi\}$

Ans: $\pi - 2$

/3 (b) $\iint_R 2y\,dA$ where R is the region enclosed between $y = 2x^2$ and $y = 1 + x^2$

Ans: 32/15