MATH 2650 Homework #3 Solutions

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§**14.7, #9.** Find the local max and min values and saddle points of $f(x,y) = x^2 + y^4 + 2xy$.

First find the critical points:

$$\begin{cases} f_x = 2x + 2y = 0 \\ f_y = 4y^3 + 2x = 0 \end{cases} \implies y = -x \implies 4(-x)^3 + 2x = 0 = 2x(1-2x^2) \implies x = 0 \text{ or } x = \pm\frac{1}{\sqrt{2}}$$

So the critical points are $(0,0)$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. To classify these using the 2nd derivative test we need to evaluate:

$$\begin{cases} f_{xx} = 2 \\ f_{yy} = 12y^2 \\ f_{xy} = 2 \end{cases} \implies D = f_{xx}f_{yy} - [f_{xy}]^2 = (2)(12y^2) - (2)^2 = 24y^2 - 4.$$

At $(0,0)$ we have $D = -4 < 0$ so this is a saddle point.
At $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ we have $D = 8 > 0$ and $f_{xx} = 2 > 0$ so $f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\frac{1}{4}$ is a local min.
At $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ we have $D = 8 > 0$ and $f_{xx} = 2 > 0$ so $f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = -\frac{1}{4}$ is a local min.

§**14.8, #9.** Use Lagrange multipliers to find the extreme values of $f(x,y,z) = xy^2z$ subject to $x^2 + y^2 + z^2 = 4$.

Let $g(x,y,z) = x^2 + y^2 + z^2$. Then

$$\nabla f = (y^2z, 2xyz, xy^2)$$
$$\nabla g = (2x, 2y, 2z).$$

So we need to solve the following system for unknowns x, y, z, λ:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases} \implies \begin{cases} y^2 z = \lambda 2x \\ 2xyz = \lambda 2y \implies 2y(xz - \lambda) = 0 \implies y = 0 \text{ or } \lambda = xz \\ xy^2 = \lambda 2z \\ x^2 + y^2 + z^2 = 4 \end{cases}$$

Case $y = 0$**:**

$$0 = \lambda 2x = \lambda 2z \implies \lambda = 0 \text{ or } x = z = 0.$$

If $x = y = z = 0$ then $x^2 + y^2 + z^2 = 4$ is not satisfied so we reject this solution. On the other hand, taking $\lambda = 0$ satisfies the first 3 equations (which correspond in this case to $\nabla f = 0$, i.e. a critical point of f). The 4th equation is then satisfied on the circle $x^2 + z^2 = 4$, $y = 0$ on which $f(x,y,z) = 0$.

Case $\lambda = xz$:

$$\begin{cases} y^2 z = (xz)2x \\ xy^2 = (xz)2z \end{cases} \implies \begin{cases} z(y^2 - 2x^2) = 0 \\ x(y^2 - 2z^2) = 0 \end{cases}$$

Satisfying the first equation with $z = 0$ gives either

$$\begin{cases} x = z = 0 \\ y = \pm 2 \\ f(x,y,z) = f(0,\pm 2,0) = 0 \end{cases} \quad \text{or} \quad \begin{cases} y = z = 0 \\ x = \pm 2 \\ f(x,y,z) = f(\pm 2,0,0) = 0. \end{cases}$$

Alternatively, satisfying the first equation with $y^2 = 2x^2$ gives either

$$\begin{cases} y = x = 0 \\ z = \pm 2 \\ f(x,y,z) = f(0,0,\pm 2) = 0. \end{cases} \quad \text{or} \quad 2z^2 = y^2 = 2x^2.$$

Imposing $x^2 + y^2 + z^2 = 4$ with the second alternative here gives

$$4 = x^2 + 2x^2 + x^2 = 4x^2 \implies x = \pm 1,\ y = \pm\sqrt{2},\ z = \pm 1.$$

We have

$$\begin{aligned} f(1,\pm\sqrt{2},1) &= 2 \\ f(1,\pm\sqrt{2},-1) &= -2 \\ f(-1,\pm\sqrt{2},1) &= -2 \\ f(-1,\pm\sqrt{2},-1) &= 2. \end{aligned}$$

So the extreme values of f are 2 (at the four points $(1,\pm\sqrt{2},1)$ and $(-1,\pm\sqrt{2},-1)$) and -2 (at the four points $(1,\pm\sqrt{2},-1)$ and $(-1,\pm\sqrt{2},1)$).

§**14.8, #17.** Find the extreme values of $f(x,y,z) = x + y + z$ subject to both $x^2 + z^2 = 2$ and $x + y = 1$.

Let $g_1(x,y,z) = x^2 + z^2$ and $g_2(x,y,z) = x + y$. Then

$$\begin{aligned} \nabla f &= (1,1,1) \\ \nabla g_1 &= (2x,0,2z) \\ \nabla g_2 &= (1,1,0) \end{aligned}$$

so we need to solve the following system for unknowns x, y, z, λ:

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 2 \\ g_2 = 1 \end{cases} \implies \begin{cases} 1 = \lambda_1(2x) + \lambda_2(1) \\ 1 = \lambda_1(0) + \lambda_2(1) \implies \lambda_2 = 1 \\ 1 = \lambda_1(2z) + \lambda_2(0) \\ x^2 + z^2 = 2 \\ x + y = 1. \end{cases}$$

This reduces to

$$\begin{cases} 0 = 2\lambda_1 x \implies \lambda_1 = 0 \text{ or } x = 0 \\ 1 = 2\lambda_1 z \\ x^2 + z^2 = 2 \\ x + y = 1. \end{cases}$$

The case $\lambda_1 = 0$ contradicts the second equation so we must have $x = 0$. This gives $y = 0$ and $z = \pm\sqrt{2}$. The extreme values of f are then

$$f(0,0,\sqrt{2}) = \sqrt{2} \quad \text{(max)}$$
$$f(0,0,-\sqrt{2}) = -\sqrt{2} \quad \text{(min)}.$$