MATH 2200 – Problem Set #3 Solutions

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Question 5

Classify each set as open, closed, both, or neither.

(a) $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ is neither open nor closed:

S is not open because $\operatorname{int} S = \emptyset \neq S$. (For every $x = \frac{1}{n} \in S$, every neighborhood $N(\frac{1}{n}, \varepsilon)$ contains irrational numbers (i.e. numbers not in S) so x is not an interior point. Thus $\operatorname{int} S = \emptyset$.)

S is not closed because 0 is a boundary point, but $0 \notin S$, so $\operatorname{bd} S \nsubseteq S$.

(b) $\mathbb{N}$ is closed but not open:

At each $n \in \mathbb{N}$, every neighbourhood $N(n, \varepsilon)$ intersects both $\mathbb{N}$ and $\mathbb{N}^{\mathrm{C}}$, so $\mathbb{N} \subseteq \operatorname{bd} \mathbb{N}$. There are no other boundary points, so in fact $\mathbb{N} = \operatorname{bd} \mathbb{N}$, so $\mathbb{N}$ is closed.

$\mathbb{N}$ is not open because $\operatorname{int} N = \emptyset \neq \mathbb{N}$. (For every $n \in N$, every neighborhood $N(n, \varepsilon)$ contains non-integer reals (i.e. numbers not in $\mathbb{N}$) so n is not an interior point. Thus $\operatorname{int} N = \emptyset$.)

(c) $\mathbb{Q}$ is neither open nor closed:

$\mathbb{Q}$ is not closed because $\operatorname{bd} \mathbb{Q} = \mathbb{R} \nsubseteq \mathbb{Q}$. (For each $x \in \mathbb{R}$, every neighborhood $N(x, \varepsilon)$ contains both rationals and irrationals, so x is a boundary point. Thus $\operatorname{bd} \mathbb{Q} = \mathbb{R}$.)

$\mathbb{Q}$ is not open because $\operatorname{int} \mathbb{Q} = \emptyset \neq \mathbb{Q}$. (For every $q \in S$, every neighborhood $N(q, \varepsilon)$ contains irrational numbers (i.e. numbers not in $\mathbb{Q}$) so q is not an interior point. Thus $\operatorname{int} \mathbb{Q} = \emptyset$.)

(d) $S = \bigcap_{n \in \mathbb{N}} \left(0, \frac{1}{n}\right)$

We have

$$S = \bigcap_{n \in \mathbb{N}} \left(0, \frac{1}{n}\right) = (0,1) \cap (0, \tfrac{1}{2}) \cap (0, \tfrac{1}{3}) \cap \cdots = \emptyset$$

so S is both open and closed (see Practice 3.4.9 in text).

(To see this, consider an arbitrary $x \in \mathbb{R}$. If $x \leq 0$ then clearly $x \notin S$ since $x \notin (0,1)$. On the other hand, if $x > 0$ then $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < x$ (by the Archimedean property) so $x \notin \left(0, \frac{1}{n}\right)$ and again $x \notin S$. Thus $S = \emptyset$.)

(e) $S = \left\{ x : \left|x - 5\right| \leq \frac{1}{2} \right\}$

S is closed but not open:

S is closed because

$$S = \left\{ x : \left|x - 5\right| \leq \frac{1}{2} \right\} = \left[4.5, 5.5\right] \operatorname{bd} S = \{4.5, 5.5\} \implies \operatorname{bd} S \subseteq S.$$

S is not open because $4.5 \in S$ is not an interior point so $S \neq \operatorname{int} S$.

(f) $S = \{x : x^2 > 0\}$

S is open but not closed:

We have $S = \mathbb{R} \setminus \{0\}$. S is open because $\operatorname{bd} S = \{0\} \in S^{\mathrm{C}}$.

S is not closed because $\operatorname{bd} S = \{0\} \not\subseteq S$.

Question 7

Let S, $T \subseteq \mathbb{R}$. Find a counter-example for each of the following:

(a) If P is the set of all isolated points of S, then P is a closed set.

Consider $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Every $x = \frac{1}{n} \in S$ is an isolated point since for sufficiently small ε (e.g. $\varepsilon < 1/(2n)$) we have $N^*(x, \varepsilon) \cap S = \emptyset$, so x is not an accumulation point. Therefore $P = S$.

From 5(a) we have that S is not closed, so P is *not* closed.

(b) Every open set contains at least two points.

Consider $S = \emptyset$ which is open (see Practice 3.4.9 in text) but contains no points.

(c) If S is closed, then $\operatorname{cl}(\operatorname{int} S) = S$.

Consider $S = \mathbb{N}$ which is closed (see 5(b)). Every point of S is a boundary point, so $\operatorname{int} S = \emptyset$. Thus

$$\operatorname{cl}(\operatorname{int} S) = \operatorname{cl}(\emptyset) = \emptyset \neq S.$$

(d) If S is open, then $\operatorname{int}(\operatorname{cl} S) = S$.

Consider $S = \mathbb{R} \setminus \{0\}$ which is open (since $\operatorname{bd} S = \{0\} \subseteq S^{\mathrm{C}}$). Because $\{0\}$ is an accumulation point of S we have $\operatorname{cl} S = \mathbb{R}$, so that

$$\operatorname{int}(\operatorname{cl} S) = \operatorname{int}(\mathbb{R}) = \mathbb{R} \neq S.$$

(e) $\operatorname{bd}(\operatorname{cl} S) = \operatorname{bd} S$.

Let $S = \mathbb{Q}$. Then

$$\begin{aligned} \operatorname{bd} S &= \mathbb{R} \\ \operatorname{cl} S &= \mathbb{R} \end{aligned}$$

so we have

$$\operatorname{bd}(\operatorname{cl} S) = \operatorname{bd}(\mathbb{R}) = \emptyset \neq \mathbb{R} = \operatorname{bd} S.$$

(f) $\operatorname{bd}(\operatorname{bd} S) = \operatorname{bd} S$.

Let $S = \mathbb{Q}$. Then

$$\operatorname{bd} S = \mathbb{R}$$

so we have

$$\operatorname{bd}(\operatorname{bd} S) = \operatorname{bd}(\mathbb{R}) = \emptyset \neq \mathbb{R} = \operatorname{bd} S.$$

(g) $\operatorname{bd}(S \cup T) = \operatorname{bd}(S) \cup \operatorname{bd}(T)$.

Consider

$$S = (0, 2), \qquad T = (1, 3).$$

Then

$$\operatorname{bd}(S \cup T) = \operatorname{bd}(0, 3) = \{0, 3\} \neq \operatorname{bd} S \cup \operatorname{bd} T = \{0, 2\} \cup \{1, 3\} = \{0, 1, 2, 3\}$$

(h) $\operatorname{bd}(S \cap T) = \operatorname{bd} S \cap \operatorname{bd} T$.

Let $S = (0, 2)$, $T = (1, 3)$. Then

$$\operatorname{bd}(S \cap T) = \operatorname{bd}(1, 2) = \{1, 2\} \neq \operatorname{bd} S \cap \operatorname{bd} T = \{0, 2\} \cap \{1, 3\} = \emptyset$$

Question 11

Show that if A is open and B is closed, that $A \setminus B$ is open and $B \setminus A$ is closed.

Proof. Suppose A is open and B is closed.

We have that B^{C} is open (Thm. 3.4.7a). Therefore

$$A \setminus B = A \cap B^{\mathrm{C}}$$

is open, since it is the intersection of a finite collection of open sets (Thm. 3.4.10b).

Furthermore, we have that A^{C} is closed (Thm. 3.4.7b). Therefore

$$B \setminus C = B \cap C^{\mathrm{C}}$$

is closed, since it is the intersection of closed sets (Cor. 3.4.11a). ■