MATH 2200 – Problem Set #3 Solutions
8 Mar. 2017

Question 5

Classify each set as open, closed, both, or neither.

(a) \( S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \) is neither open nor closed:

\( S \) is not open because \( \text{int} \, S = \emptyset \neq S \). (For every \( x = \frac{1}{n} \in S \), every neighborhood \( N(\frac{1}{n}, \varepsilon) \) contains irrational numbers (i.e. numbers not in \( S \)) so \( x \) is not an interior point. Thus \( \text{int} \, S = \emptyset \).)

\( S \) is not closed because 0 is a boundary point, but 0 \( \notin \) \( S \), so \( \text{bd} \, S \notin S \).

(b) \( \mathbb{N} \) is closed but not open:

At each \( n \in \mathbb{N} \), every neighbourhood \( N(n, \varepsilon) \) intersects both \( \mathbb{N} \) and \( \mathbb{N}^C \), so \( \mathbb{N} \subseteq \text{bd} \, \mathbb{N} \), so \( \mathbb{N} \) is closed.

\( \mathbb{N} \) is not open because \( \text{int} \, \mathbb{N} = \emptyset \neq \mathbb{N} \). (For every \( n \in \mathbb{N} \), every neighborhood \( N(n, \varepsilon) \) contains non-integer reals (i.e. numbers not in \( \mathbb{N} \)) so \( n \) is not an interior point. Thus \( \text{int} \, \mathbb{N} = \emptyset \).)

(c) \( \mathbb{Q} \) is neither open nor closed:

\( \mathbb{Q} \) is not closed because \( \text{bd} \, \mathbb{Q} = \mathbb{R} \neq \mathbb{Q} \). (For each \( x \in \mathbb{R} \), every neighborhood \( N(x, \varepsilon) \) contains both rationals and irrationals, so \( x \) is a boundary point. Thus \( \text{bd} \, \mathbb{Q} = \mathbb{R} \).)

\( \mathbb{Q} \) is not open because \( \text{int} \, \mathbb{Q} = \emptyset \neq \mathbb{Q} \). (For every \( q \in \mathbb{S} \), every neighborhood \( N(q, \varepsilon) \) contains irrational numbers (i.e. numbers not in \( \mathbb{Q} \)) so \( q \) is not an interior point. Thus \( \text{int} \, \mathbb{Q} = \emptyset \).)

(d) \( S = \bigcap_{n \in \mathbb{N}} \left( 0, \frac{1}{n} \right) \)

We have

\[ S = \bigcap_{n \in \mathbb{N}} \left( 0, \frac{1}{n} \right) = (0, 1) \cap (0, \frac{1}{2}) \cap (0, \frac{1}{3}) \cap \cdots = \emptyset \]

so \( S \) is both open and closed (see Practice 3.4.9 in text).

(To see this, consider an arbitrary \( x \in \mathbb{R} \). If \( x \leq 0 \) then clearly \( x \notin S \) since \( x \notin (0, 1) \). On the other hand, if \( x > 0 \) then \( \exists n \in \mathbb{N} \) such that \( \frac{1}{n} < x \) (by the Archimedean property) so \( x \notin \left( 0, \frac{1}{n} \right) \) and again \( x \notin S \). Thus \( S = \emptyset \).)

(e) \( S = \left\{ x : |x - 5| \leq \frac{1}{2} \right\} \)

\( S \) is closed but not open:

\( S \) is closed because

\[ S = \left\{ x : |x - 5| \leq \frac{1}{2} \right\} = [4.5, 5.5] \text{ bd } S = \{4.5, 5.5\} \implies \text{bd} \, S \subseteq S. \]
(f) $S = \{x : x^2 > 0\}$

S is open but not closed:

We have $S = \mathbb{R} \setminus \{0\}$. $S$ is open because $\text{bd} S = \{0\} \in S^C$.

$S$ is not closed because $\text{bd} S = \{0\} \not\subseteq S$.

**Question 7**

Let $S, T \subseteq \mathbb{R}$. Find a counter-example for each of the following:

(a) If $P$ is the set of all isolated points of $S$, then $P$ is a closed set.

Consider $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Every $x = \frac{1}{n} \in S$ is an isolated point since for sufficiently small $\varepsilon$ (e.g. $\varepsilon < 1/(2n)$) we have $N^*(x, \varepsilon) \cap S = \emptyset$, so $x$ is not an accumulation point. Therefore $P = S$.

From 5(a) we have that $S$ is not closed, so $P$ is not closed.

(b) Every open set contains at least two points.

Consider $S = \emptyset$ which is open (see Practice 3.4.9 in text) but contains no points.

(c) If $S$ is closed, then $\text{cl}(\text{int} S) = S$.

Consider $S = \mathbb{N}$ which is closed (see 5(b)). Every point of $S$ is a boundary point, so $\text{int} S = \emptyset$. Thus

$$\text{cl}(\text{int} S) = \text{cl}(\emptyset) = \emptyset \neq S.$$  

(d) If $S$ is open, then $\text{int}(\text{cl} S) = S$.

Consider $S = \mathbb{R} \setminus \{0\}$ which is open (since $\text{bd} S = \{0\} \subseteq S^C$). Because $\{0\}$ is an accumulation point of $S$ we have $\text{cl} S = \mathbb{R}$, so that

$$\text{int}(\text{cl} S) = \text{int}(\mathbb{R}) = \mathbb{R} \neq S.$$  

(e) $\text{bd}(\text{cl} S) = \text{bd} S$.

Let $S = \mathbb{Q}$. Then

$$\text{bd} S = \mathbb{R} \quad \text{cl} S = \mathbb{R}$$

so we have

$$\text{bd}(\text{cl} S) = \text{bd}(\mathbb{R}) = \emptyset \neq \mathbb{R} = \text{bd} S.$$  

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(f) \( \text{bd}(\text{bd} S) = \text{bd} S \).

Let \( S = \mathbb{Q} \). Then 
\[
\text{bd} S = \mathbb{R}
\]
so we have 
\[
\text{bd}(\text{bd} S) = \text{bd}(\mathbb{R}) = \emptyset \neq \mathbb{R} = \text{bd} S.
\]

(g) \( \text{bd}(S \cup T) = \text{bd}(S) \cup \text{bd}(T) \).

Consider
\[
S = (0, 2), \quad T = (1, 3).
\]
Then
\[
\text{bd}(S \cup T) = \text{bd}(0, 3) = \{0, 3\} \neq \text{bd} S \cup \text{bd} T = \{0, 2\} \cup \{1, 3\} = \{0, 1, 2, 3\}
\]

(h) \( \text{bd}(S \cap T) = \text{bd} S \cap \text{bd} T \).

Let \( S = (0, 2), \ T = (1, 3) \). Then
\[
\text{bd}(S \cap T) = \text{bd}(1, 2) = \{1, 2\} \neq \text{bd} S \cap \text{bd} T = \{0, 2\} \cap \{1, 3\} = \emptyset
\]

**Question 11**

Show that if \( A \) is open and \( B \) is closed, that \( A \setminus B \) is open and \( B \setminus A \) is closed.

**Proof.** Suppose \( A \) is open and \( B \) is closed.

We have that \( B^C \) is open (Thm. 3.4.7a). Therefore
\[
A \setminus B = A \cap B^C
\]
is open, since it is the intersection of a finite collection of open sets (Thm. 3.4.10b).

Furthermore, we have that \( A^C \) is closed (Thm. 3.4.7b). Therefore
\[
B \setminus C = B \cap C^C
\]
is closed, since it is the intersection of closed sets (Cor. 3.4.11a).