Thompson Rivers University

Calculus I

Lecture Notes for MATH 114

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About this Document

These lecture notes are intended for my use only, and they are very much a work in progress. I make them available for free and without any warranty, expressed or implied. In fact you probably shouldn't use them at all.

This is not a textbook. Please, buy the required textbook for the course and use it. My aim is to produce a condensed set of notes to help me organize and improve upon my lecture material. Thus, very few of my examples include full solutions; these are to be worked out in class, along with the sundry other details, diagrams, humorous anecdotes, and everything else that makes attending lectures worthwhile. Do not use these as a substitute for taking notes in class.

Calculus I

Lecture Notes for MATH 114

Course Syllabus

Section numbers refer to the assigned text: James Stewart, *Single Variable Calculus: Concepts and Contexts*, 3rd edition, Thomson, 2006.

1. **Functions**
   domain and range 1.1
   sketching & manipulating graphs 1.1
   composition/decomposition of functions 1.3

2. **Calculus motivation**
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   parametric curves 3.5

1 Preliminaries

Lec. #1

- administer Calculus Preparedness Test

Lec. #2

- mark test and begin discussing
- finish discussing Calculus Preparedness Test

2 Functions

2.1 Preliminary Discussion

What is a function?

- *rule* that says how to calculate output given input
- a *transformation* of (i.e. action on) a number

Functions are everywhere. In the sciences, almost all of what you learn or research is about relationships between quantities (e.g. pressure and temperature in an ideal gas, carrying capacity and population growth, genetic information and the form/function of an organism, ...). These relationships can be expressed as functions. Often we don't know a formula for the relationship, but nevertheless it is the *relationship* (hence a function) that is the main object of study.

Ways to specify functions

- formula $f(x) = \cdots$
- table of values (not very detailed)

Lec. #3

- graph (not very precise)

Example 2.1. Express the surface area of a sphere as a function of its circumference, C.

Example 2.2. A box is to be designed so it has a square base and an open top, with a volume of $100\,\text{cm}^3$. Express the surface area as a function of the width x of the base.

2.2 Domain and Range

Definition 2.1. For a given function $f(x)$ the **domain** of f, written $D(f)$, is the set of all numbers x such that $f(x)$ is defined.

Definition 2.2. For a given function $f(x)$ the **range** of f, written $R(f)$, is the set of all numbers y such that $y = f(x)$ for some number x in the domain of f.

Example 2.3. Find the domain range of: $f(x) = x^2 \quad g(x) = \sqrt{x} \quad h(x) = \sqrt{1-x^2}$

2.3 Functions whose graphs you should know

x^2, x^3, x^4, $\frac{1}{x}$, $\frac{1}{x^2}$, $\sin x$, $\cos x$, $\tan x$, e^x, $\ln x$

2.4 Transformations of graphs

Given graph of $f(x)$, how to graph $f(x-a)$, $f(x+a)$, $af(x)$, $f(ax)$.

3 The Derivative and Limits: Motivation

Lec. #4

3.1 Instantaneous Velocity

Basic Problem: Suppose an object is moving along the x-axis, and that its *position* at any time t is given by a function $x = f(t)$. How do we find a function that gives the object's *velocity* at time t?

Example 3.1. Suppose $f(t) = t^2$. How to find the velocity at time $t = 3$?

Make table of values for $x = f(t)$ for t near 3. Consider *average* velocity over some time interval starting at $t = 3$.

The velocity we get depends on the averaging interval Δt; smaller Δt gives more accurate velocity. But what is the *exact*, *instantaneous* velocity at $t = 3$? Clearly it shouldn't rely on any particular value of Δt.

Idea: as Δt gets smaller, the average v we calculate gets closer and closer to some value (which we suspect is 6)... this value gives the *instantaneous* velocity. We can justify the velocity 6 more rigorously

Notation: $\lim_{\Delta t \to 0}(\cdots)$ means "the limiting value, as Δt gets closer and closer to 0, of the expression $(\cdots)$". Then at $t = 2$ the velocity is

$$\begin{aligned} v &= \lim_{\Delta t \to 0} \frac{(3+\Delta t)^2 - (3)^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{9 + 6\Delta t + (\Delta t)^2 - 9}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{6\Delta t + (\Delta t)^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} (6 + \Delta t) \\ &= 6. \end{aligned}$$

In general, the velocity at time t is given by

$$v(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

3.2 Slopes of Graphs

Basic Problem: Given a function $f(x)$, how do we find the slope of the graph (i.e. the slope of the *tangent line* to the graph) at a particular point?

Example 3.2. Suppose $f(x) = x^2$. How to find the slope at the point where $x = 3$?

Start with something close to the tangent line: the *secant* line joining $(x, f(x))$ to a nearby point $(x+h, f(x+h))$:

$$m = \frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{h}$$

Of course the slope we get depends on h. In our case, with $f(x) = x^2$ and $x = 2$:

$$\begin{aligned} m &= \frac{f(2+h)-f(2)}{h} \\ &= \frac{(2+h)^2-2^2}{h} \\ &= \frac{4+4h+h^2-4}{h} \\ &= 4+h \end{aligned}$$

So with $h = 0.1$ we get a slope $m = 4.1$. With $h = 0.01$ we get $m = 4.01$. As h gets smaller, the secant line gets closer to the tangent line. In fact, the slope of the tangent line is the *limiting value* that the slope of the secant line approaches as h goes to 0:

$$m = \lim_{h\to 0}(4+h) = 4$$

4 Limits

Lec. #5

This course is really about the derivative, but to calculate derivatives we need to be able to understand and calculate limits, like the $\lim_{h\to 0}$ that appears in the definition.

Notation: $\lim\limits_{x\to a} f(x) = L$ means, roughly, "as x approaches the number a the value of $f(x)$ approaches the number L."

4.1 Intuitive & Graphical Approach to Limits

Example 4.1. $\lim\limits_{x\to 3} x^2 = 9$ (interpret in terms of graph)

- graphical example for a function with a hole

- graphical example for a function with a point discontinuity

- graphical example for a function with a jump discontinuity: the limit doesn't exist

Notation: $\lim\limits_{x\to a^-} f(x)$ means "the limit as $x \to a$ from the left". Similarly, $\lim\limits_{x\to a^+} f(x)$ means "the limit as $x \to a$ from the right".

Theorem 4.1. *The limit* $\lim\limits_{x\to a} f(x)$ *exists and equals* L *if and only if* $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = L$.

Example 4.2. Evaluate $\lim_{x\to 5} f(x)$, if it exists, for the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 5 \\ 25 - x & \text{if } x > 5 \end{cases}.$$

(interpret in terms of the graph)

4.2 Algebraic Tricks for Computing Limits

Common tricks involving factoring, reciprocals and roots:

Example 4.3. Evaluate (a) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$ (b) $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x-3}$

Example 4.4. Evaluate (a) $\lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{x+6}-3}$ (b) $\lim\limits_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}$

Example 4.5. Evaluate (a) $\lim\limits_{x\to 4}\dfrac{\frac{1}{x}-\frac{1}{4}}{x-4}$ (b) $\lim\limits_{x\to 1}\dfrac{\frac{1}{x+2}-\frac{1}{3}}{x-1}$

Lec. #6

4.3 Limit Laws

Guessing values of limits isn't good enough; we need some rules.

Suppose that $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist; then:

1. $\lim\limits_{x\to a} c = c$
2. $\lim\limits_{x\to a} x^n = a^n$
3. $\lim\limits_{x\to a} [f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$
4. $\lim\limits_{x\to a} [cf(x)] = c\lim\limits_{x\to a} f(x)$
5. $\lim\limits_{x\to a} [f(x)g(x)] = \left[\lim\limits_{x\to a} f(x)\right]\cdot\left[\lim\limits_{x\to a} g(x)\right]$
6. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ provided $\lim\limits_{x\to a} g(x) \neq 0$.
7. $\lim\limits_{x\to a} [f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$
8. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$

Example 4.6. In a previous example we had to evaluate

$$\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}\,\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4}\frac{(x-4)(\sqrt{x}+2)}{x-4} = \lim_{x\to 4}(\sqrt{x}+2).$$

Of course we suspect the answer is $\sqrt{4}+2=4$, but we need to justify this using the limit laws:

$$\begin{aligned}
\lim_{x\to 4}(\sqrt{x}+2) &= \lim_{x\to 4}\sqrt{x} + \lim_{x\to 4} 2 && \text{(rule 3)}\\
&= \sqrt{\lim_{x\to 4} x} + 2 && \text{(rules 1 \& 8)}\\
&= \sqrt{4}+2 && \text{(rule 2)}\\
&= 4.
\end{aligned}$$

4.4 Continuity

- intuitive graphical examples (where $f(a)$ dne, or the limit dne, or both)

Definition 4.1. A function $f(x)$ is continuous at $x=a$ if $\lim\limits_{x\to a} f(x) = f(a)$.

- justify our conclusions in previous examples

Example 4.7. Let $f(x) = \begin{cases} 1-x & \text{if } x<1 \\ x^2 & \text{if } x\geq 1\end{cases}$. Is $f(x)$ continuous at $x=1$?

Example 4.8. Let $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x > 0 \\ C & \text{if } x = 0. \end{cases}$. For what value(s) of C is $f(x)$ continuous at $x = 0$?

The following functions are continuous everywhere on their domains: x^2, x^3, ..., $\sin x$, $\cos x$, $\tan x$, $1/x$, $\sqrt{x}$, e^x, $\ln x$, ... in fact every elementary function you know.

Theorem 4.2. *Suppose $f(x)$ and $g(x)$ are both continuous at $x = a$. Then the following are also continuous at $x = a$:*

1. $f(x) \pm g(x)$
2. $f(x)g(x)$
3. $\dfrac{f(x)}{g(x)}$ *(provided $g(a) \neq 0$)*
4. $(f \circ g)(x)$ *(provided $f(x)$ is continuous at $x = g(a)$)*

Example 4.9. At what value(s) of x are the following *not* continuous?
(a) $f(x) = \dfrac{\sin x}{x}$ (b) $g(x) = \dfrac{x+1}{x^2 + x - 2}$ (c) $h(x) = \dfrac{\tan x}{x - 1}$ (d) $F(x) = \dfrac{x}{\sin x}$

4.5 Limits Involving Infinity

Lec. #7

Up to this point, we would have said that $\lim_{x\to 0} \frac{1}{x^2}$ does not exist. However, it's more precise to say that

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty,$$

in that the function $1/x^2$ gets arbitrarily large (and positive) as x approaches 0 from either the left or the right. Similarly, we have

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

The following example introduces a notation that is helpful in evaluating limits involving infinity.

Example 4.10. Evaluate $\displaystyle\lim_{x\to 3^+} \frac{(x+2)(x-5)}{(x+1)(3-x)}$.

Strictly speaking our limits laws do not apply, because the denominator has a limit of zero. However, it is help to write:

$$\lim_{x\to 3^+} \frac{(x+2)(x-5)}{(x+1)(3-x)} = \frac{(5)(-2)}{(4)(0^-)} = +\infty.$$

Here the symbol 0^- is used to represent the *negative* quantity $3 - x$, which approaches 0 from the left as $x \to 3^+$. Overall the function evaluates to a *positive* quantity (because of the -2 in the numerator) that becomes arbitrarily large as $x \to 3^+$, hence the limiting value if $+\infty$.

Example 4.11. Evaluate: (a) $\displaystyle\lim_{x\to 2^+} \frac{5}{2-x}$ (b) $\displaystyle\lim_{x\to 0^-} \frac{5x(2+x)}{x(3-x)}$ (c) $\displaystyle\lim_{x\to -3^+} \frac{4x + x^2}{9 - x^2}$

Another kind of limit involving infinity is the following:

$$\lim_{x\to\infty} \frac{1}{x}$$

which means "the limiting value of the function $1/x$ as x becomes arbitrarily large (and positive)". Thus $\lim_{x\to\infty} \frac{1}{x} = 0$.

The following example illustrates a useful trick in evaluating this kind of limit.

Example 4.12. Evaluate $\lim_{x\to\infty} \frac{x+2}{x^2+8}$.

We can rewrite the limit as follows, by dividing numerator and denominator by the *highest* power of x:

$$\lim_{x\to\infty} \frac{x+2}{x^2+8} = \lim_{x\to\infty} \frac{(x+2)\,\frac{1}{x^2}}{(x^2+8)\,\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{2}{x^2}}{1+\frac{8}{x^2}}.$$

Now apply limit laws:

$$\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{2}{x^2}}{1+\frac{8}{x^2}} = \frac{\lim\limits_{x\to\infty}\left(\frac{1}{x}+\frac{2}{x^2}\right)}{\lim\limits_{x\to\infty}\left(1+\frac{8}{x^2}\right)} = \frac{\lim\limits_{x\to\infty}\frac{1}{x} + \lim\limits_{x\to\infty}\frac{2}{x^2}}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty}\frac{8}{x^2}} = \frac{0+0}{1+0} = 0.$$

Example 4.13. Evaluate: (a) $\lim_{x\to\infty} \frac{2x+3}{x-8}$ (b) $\lim_{x\to-\infty} \frac{x^3+3x+1}{2x^3-x^2}$ (c) $\lim_{x\to\infty} \frac{x^2}{x+1}$

5 Derivatives

5.1 Definition of the Derivative

Lec. #8

In both instantaneous velocity and slopes of tangent lines, we ended up with an expression like:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

(In the velocity problem, the x's were t's, and h was a Δt, but in form the expressions are the same.)

This expression arises in so many different applications that is has a name, and a shorthand notation:

Definition 5.1. The **derivative** of a function $f(x)$ is given by

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

and is defined for all x for which this limit exists.

Note that $f'(x)$ is itself a function of x. For a given number x, the number $f'(x)$ can be interpreted as a formula for the slope of the tangent line to the graph $y = f(x)$ at x.

It is important to *memorize* this definition and to *understand* where it comes from.

Notation: The following alternative notations for the derivative are useful in different situations:

$$\boxed{f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x)}$$

The symbol $\frac{d}{dx}$ means "the derivative (with respect to x) of the expression that follows".

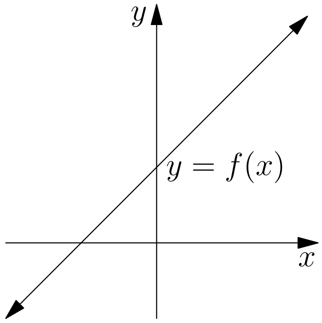
5.2 Calculating Derivatives Using the Definition

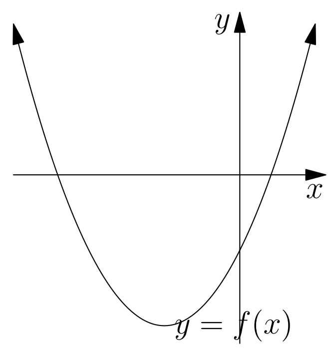
Example 5.1. Use the definition of the derivative to find $f'(x)$ for:
(a) $f(x) = x^2$ (b) $f(x) = 3x$ (c) $f(x) = \frac{1}{x}$ (d) $f(x) = \sqrt{x}$

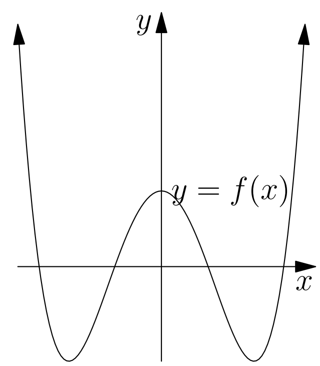
Example 5.2. For each function in the previous example, find the slope of $y = f(x)$ at $x = 4$.

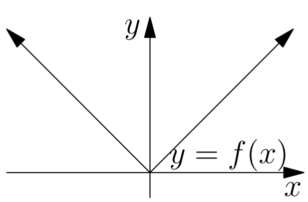
5.3 Derivative Functions from Graphs

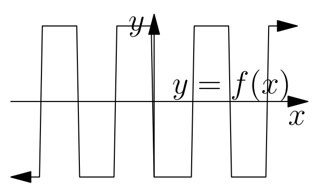
Sketch some graphs of $y = f(x)$, and from these sketch graphs of $y = f'(x)$.

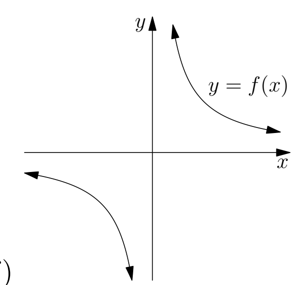
(a)


(b)


(c)


(d)


(e)


(f)


Lec. #9

5.4 Power Rule

Calculating derivatives using the definition can be very difficult. Consider finding $f'(x)$ for the complicated function $f(x) = \sqrt{\cos^2(\ln x) + \sec x}$; eventually we'll see how, but it's much too hard to do it directly from the definition. Instead, we'll use some general properties of the derivative that allow us avoid the limit calculation.

We already used the definition to prove the $\frac{d}{dx}x^2 = 2x$. Similar calculations show that $\frac{d}{dx}x^3 = 3x^2$, $\frac{d}{dx}x^4 = 4x^3$, and in fact we could prove the general rule that

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

So, for example, we now say (without going to the trouble of using the definition of the derivative) that $\frac{d}{dx}x^1 0 = 10x^9$.

5.5 Derivative Rules

From now on we will calculate derivatives symbolically, by recognizing patterns and using the following symbolic rules, rather than use the definition directly.

1. $\dfrac{d}{dx}C = 0$

2. $\dfrac{d}{dx}x^n = nx^{n-1}$ [power rule]

3. $\dfrac{d}{dx}\big(Cf(x)\big) = Cf'(x)$

4. $\dfrac{d}{dx}\big[f(x) \pm g(x)\big] = f'(x) \pm g'(x)$

5. $\dfrac{d}{dx}\big[f(x)g(x)\big] = f'(x)g(x) + f(x)g'(x)$ [product rule]

6. $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ [quotient rule]

7. $\dfrac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x)$ [chain rule]

The following examples involve applications of the first four rules.

Example 5.3. Differentiate symbolically:
(a) $\dfrac{d}{dx}(x^3 + x^5 + 2)$ (b) $\dfrac{d}{dx}5x^3$ (c) $\dfrac{d}{dx}\left(\dfrac{1}{x^2} - \dfrac{1}{x}\right)$ (d) $\dfrac{d}{dx}(x+2)^2$

Example 5.4. Find the slope of the tangent line to $y = x + \dfrac{2}{x}$ at $x = 3$.

The following examples involve the product and quotient rules.

Example 5.5. Differentiate:
(a) $\dfrac{d}{dx}(x^3 + x^5)(2 + x)$ (b) $\dfrac{d}{dx}x^2(x - 8)$ (c) $\dfrac{d}{dx}\dfrac{x^2}{x^4 - x}$ (d) $\dfrac{d}{dx}\dfrac{3x^4 + 8x^2 + 2}{x^2 - 5}$

The following examples involve the chain rule.

Example 5.6. Differentiate:
(a) $\dfrac{d}{dx}(x^3 + 2)^6$ (b) $\dfrac{d}{dx}(5x^2 - x)^{10}$ (c) $\dfrac{d}{dx}\dfrac{1}{2x - 1}$ (d) $\dfrac{d}{dx}\sqrt{3x + 2}$

Example 5.7. Find the slope of $y = \sqrt{x - 5}$ at $x = 14$.

5.6 Derivatives of Important Functions

You will need to know the derivatives of the following important elementary functions (all of the functions that are important enough to be available on a scientific calculator).

1. $\dfrac{d}{dx}\sin x = \cos x$	4. $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$
2. $\dfrac{d}{dx}\cos x = -\sin x$	5. $\frac{d}{dx}\ln x = \frac{1}{x}$
3. $\dfrac{d}{dx}\tan x = \sec^2 x$	

The following examples combine these derivative formulas with the various derivative rules.

Example 5.8. Differentiate the following:
(a) $\dfrac{d}{dx}x^2\sin x$ (b) $\dfrac{d}{dx}\dfrac{e^x}{x+2}$ (c) $\dfrac{d}{dx}\sin(x^2)$ (d) $\dfrac{d}{dx}e^{3x}$ (e) $\dfrac{d}{dx}\sqrt{xe^x}$

6 Implicit Differentiation

Lec. #10

Example 6.1. Find the derivatives of:
(a) $f(x) = x^2$ (b) $f(x) = (\sin x)^2$ (c) $f(x) = (\ln x)^2$ (d) $f(x) = (1 + x^3)^2$

Notice the pattern? Suppose $y(x)$ is a given differentiable function. Write a formula (in terms of $y'(x)$) for the derivative of $f(x) = (y(x))^2$.

Example 6.2. Suppose $y(x)$ is a given differentiable function. Write formulas (in terms of $y'(x)$) for the derivatives of:
(a) $f(x) = (y(x))^3$ (b) $f(x) = x^2y(x)$ (c) $f(x) = x^3 + y(x)$ (d) $f(x) = y(x)\sin(y(x))$

The technique above is useful for finding derivatives of *implicitly defined functions.*

Example 6.3. Suppose the function $y(x)$ satisfies $x^2 + y^2 = 10$. (The graph of $y(x)$ is a semicircle.) Find the slope of the graph of $y(x)$ at the point $(1, 3)$.

We know the function $y(x)$ satisfies the equation

$$x^2 + (y(x))^2 = 10.$$

We could solve this for $y(x)$, then differentiate. But there is an easier way. Differentiate both sides of the equation (with respect to x):

$$2x + 2y(x)y'(x) = 0 \quad \text{or simply} \quad 2x + 2yy' = 0.$$

We can solve this for $y'(x)$:

$$2yy' = -2x \implies y' = -\frac{x}{y}.$$

So, at the point $(1, 3)$, the slope is

$$y' = -\frac{1}{3}.$$

In the previous example, the function $y(x)$ is *implicitly* defined by the relationship $x^2 + y^2 = 10$. That is, the equation could, in principle, be solved for $y(x)$. Until the details are worked out and the equation is actually solved, the precise definition of $y(x)$ remains implicit. If we solve the equation we get $y(x) = \sqrt{10 - x^2}$, which is an *explicit* representation of $y(x)$.

The technique of *implicit differentiation* used above makes it possible to differentiate implicitly defined functions, without actually solving to get an explicit formula for $y(x)$.

Example 6.4. Find a formula for y' if the function $y(x)$ is defined implicitly by the equation

$$x^3 + y^3 = 1.$$

Example 6.5. Find a formula for y':
(a) $x^2 - 2xy + y^3 = 1$ (b) $y^5 + x^2y^3 = 1 + ye^x$ (c) $x\cos y + y\cos x = 1$ (d) $\sin\left(\frac{x}{y}\right) = 1$

Example 6.6. Find the slope of the curve

$$2(x^2 + y^2)^2 = 25(x^2 - y^2)$$

at the point $(3, 1)$.

Lec. #11

6.1 Inverse Trigonometric Functions

So far we have formulas for the derivatives of every important function on the scientific calculator except the inverse trigonometric functions arcsin, arccos and arctan.

Recall that $\arcsin x$ returns the angle (between $-\pi/2$ and $\pi/2$) whose sine is x. That is,

$$y = \arcsin x \iff x = \sin y.$$

The other inverse trig functions are defined similarly.

Notation: The functions $\arcsin x$, $\arccos x$ and $\arctan x$ are sometimes written $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ respectively. But be careful: while the notation $\sin^2 x$ is used to represent $(\sin x)^2$, in the special case of $\sin^{-1} x$ we do *not* mean $(\sin x)^{-1}$. Here, the -1 refers to the *inverse* of the sine function, not its reciprocal. That is,

$$\boxed{\sin^{-1} x \neq \frac{1}{\sin x}}$$

We can find the derivatives of inverse trig functions using implicit differentiation.

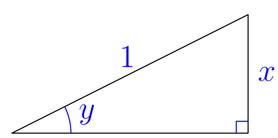
Example 6.7. To derive a formula for $\frac{d}{dx} \arcsin x$, let $y(x)$ be defined by $y = \arcsin x$. Or equivalently,

$$\sin y = x.$$

Apply implicit differentiation:

$$(\cos y)y' = 1 \implies y' = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)}.$$

This formula can be simplified using the following trick. Recall that $y = \arcsin x$ is the angle whose sine is x. This relationship between y and x is illustrated in the figure below.



To evaluate $\cos y$ we can use Pythagoras to solve for the adjacent side, $\sqrt{1-x^2}$. Then

$$\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

So we have

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

Use the same method to find the derivatives of the other inverse trig functions, and memorize them:

$$\boxed{\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \arctan x = \frac{1}{1+x^2}}$$

Lec. #12

6.2 Logarithmic Differentiation

We have no derivative rules that tell us how to differentiate something like

$$\frac{d}{dx}\left(x^{\sin x}\right)$$

because the function $\sin x$ appears in the exponent (try it and you'll see).

You can use implicit differentation to get around this problem. Let

$$y(x) = x^{\sin x}.$$

Take logarithms of both sides to get

$$\ln y = \ln(x^{\sin x})$$

then use algebraic rules for logarithms to write this as

$$\ln y = \sin x \ln(x).$$

Now apply implicit differentiation:

$$\frac{1}{y}y' = \cos x \ln(x) + (\sin x)\frac{1}{x} \implies y' = y\left[\cos x \ln(x) + (\sin x)\frac{1}{x}\right].$$

Substituting $x^{\sin x}$ back in for y we get

$$y' = x^{\sin x}\left[\cos x \ln(x) + (\sin x)\frac{1}{x}\right].$$

Example 6.8. Differentiate: (a) $\frac{d}{dx}(x^x)$ (b) $\frac{d}{dx}(2^x)$ (c) $\frac{d}{dx}(\ln x)^x$ (d) $\frac{d}{dx}(\cos x)^{\sqrt{x}}$

The previous example (b) can be generalized to the following:

$$\boxed{\frac{d}{dx}(a^x) = (a^x)\ln(a)}$$

7 Applications

7.1 Related Rates

Lec. #13

Example 7.1. Consider a circle whose radius is 10 cm and growing at 2 cm/s. At what rate is the area of the circle increasing?

If we represent the radius by an (unknown) function $r(t)$ (where t is the time in seconds) then the area as a function of time can be written

$$A(t) = \pi\big(r(t)\big)^2.$$

Use implicit differentiation to differentiate both sides with respect to t:

$$A' = \pi \cdot 2rr'.$$

Recall that A' and r' represent the instantaneous rates of change of the area and the radius, respectively. Using the given data $r = 10\,\text{cm}$ and $r' = 2\,\text{cm/s}$ we get:

$$A' = \pi \cdot 2(10\,\text{cm})(2\,\text{cm/s}) = 40\pi\,\text{cm}^2/\text{s} \approx 126\,\text{cm}^2/\text{s}$$

The previous example is a *related rates* problem in that $A(t)$ and $r(t)$ are functions of time whose rates of change are related through the equation $A = \pi r^2$. Implicit differentation is used to find the relationship relating the rates of change A' and r'.

Example 7.2. We can model the human heart as a sphere of radius 5 cm. As the heart contracts its radius decreases at about 1 cm/0.1 s= 20 cm/s. At what rate is blood expelled from the heart as it contracts?

For a sphere heart the volume is

$$V = \tfrac{4}{3}\pi r^3$$

where $V(t)$ and $r(t)$ are both (unknown) functions of time. Differentiating with respect to t we get

$$V' = \tfrac{4}{3}\pi \cdot 3r^2 \cdot r' = 4\pi r^2 r'.$$

Using $r = 5\,\text{cm}$ and $r' = -10\,\text{cm/s}$ we get

$$V' = 4\pi(5)^2(-10) \approx -3141\,\text{cm}^3/\text{s} \approx -3.1\,\text{L/s}.$$

Example 7.3. Susan is 1.5 m tall. As she walks away from a 3 m-tall lamppost at 2 m/s, what is the speed of the tip of her shadow? [*similar triangles*]

Example 7.4. The clock on the clocktower has a minute hand 80 cm long and an hour hand 40 cm long. At 3:00, at what range is the distance between the ends of the hands changing? [*cosine law*]

Example 7.5. Sand is being added to a conical pile at a rate of $5\,\text{m}^3/\text{min}$. At all times the pile retains its shape so that the cone's height is always exactly twice its radius. At what rate is the height of the pile increasing when the height is 2 m?

Lec. #14

7.2 Extreme Values

Example 7.6. A farmer has 100 m of fencing material and wants to build a rectangular enclosure for his llamas. What should the dimensions of the enclosure be, so that it has the maximum possible area?

Let the enclosure have width x and height y. The farmer needs to maximize the area

$$A = xy.$$

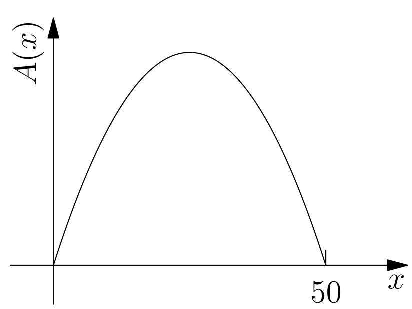
Because he has a limited amount of fencing, he can't just make both x and y as large as he wants. In fact, if he chooses a value for the width, x, then the height y is automatically determined by the remaining fencing:

$$2x + 2y = 100 \implies 2y = 100 - 2x \implies y = \frac{100 - 2x}{2} = 50 - x.$$

We can use this to eliminate y and write the area as a function of the length of just one of the sides:

$$A = A(x) = x(50 - x) = 50x - x^2.$$

The graph of this function is shown below.



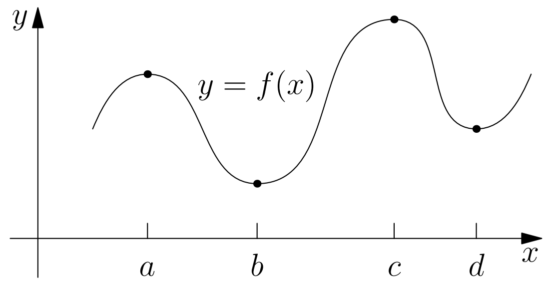
Notice: the maximum of $A(x)$ occurs where the graph has a horizontal tangent line, i.e. where $A'(x) = 0$. Using this observation, we can find the value of x that gives the maximum of $A(x)$:

$$A'(x) = 0 \implies 50 - 2x = 0 \implies x = 25.$$

So, to maximize the area of the enclosure, the dimensions should be $x = 25$ and $y = 50 - 25 = 25$ (i.e. the enclosure should be a square).

Some terminology:

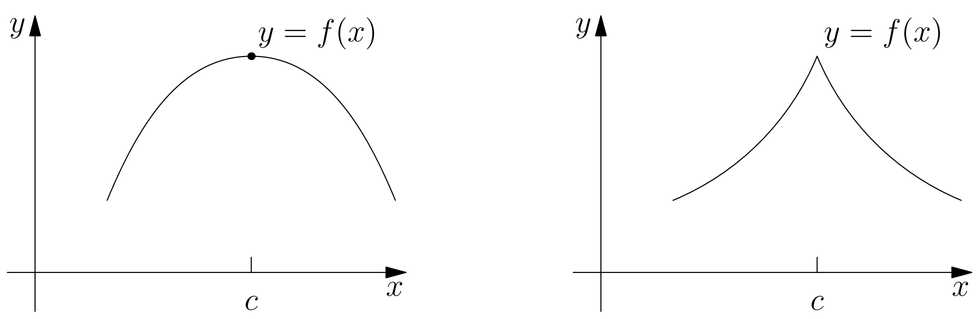
Consider the graph of $y = f(x)$ below.



- f has an *absolute* or *global maximum* at $x = c$
- f has an *absolute* or *global minimum* at $x = b$
- f has a *local maximum* at $x = a$ and at $x = c$
- f has a *local minimum* at $x = b$ and at $x = d$

Definition 7.1. A function f on in interval $[a, b]$ has an *absolute* or *global*...
...*maximum* at $x = c$ if $f(c) \geq f(x)$ for all $x \in [a, b]$.
...*minimum* at $x = c$ if $f(c) \leq f(x)$ for all $x \in [a, b]$.

There are only two ways for a function $f(x)$ to have a local extremum (i.e. a local max or min). These are illustrated below.

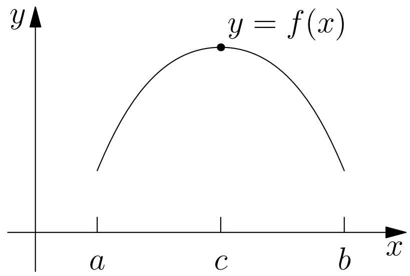


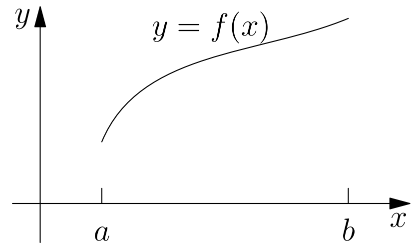
This motivates the following definition and theorem.

Definition 7.2. A number c is a critical number for a function f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 7.1. *If $f(x)$ has a local maximum or minimum at $x = c$ then c is a critical number for f.*

In applications we usually want to find an absolute extremum, not just the local extrema. If f is a continuous function on a closed interval $[a, b]$ then there are just two ways for an absolute extremum to occur: either the extremum occurs at a local extremum inside the interval, or it occurs at one of the endpoints of the interval. These two possibilities are illustrated below.





Together with the previous theorem, this observation suggests the following recipe for finding local extrema.

Closed Interval Method: To find the absolute minimum and maximum of a continuous function $f(x)$ on an interval $[a, b]$:

1. Find the critical numbers for f in the interval $[a, b]$.
2. Evaluate f at the critical numbers and at the endpoints.
3. Select the largest and the smallest values from step 2; these are the absolute maximum and minumum values of f on $[a, b]$.

Example 7.7. Find the absolute maximum and minimum values of $f(x) = x^3 - 6x^2 + 9x + 2$ on the interval $[-1, 5]$.

First we find the critical numbers for f:

$$f'(x) = 3x^2 - 12x + 9$$

so

$$\begin{aligned} f'(x) = 0 &\implies 3x^2 - 12x + 9 = 0 \\ &\implies 3(x^2 - 4x + 3) = 0 \\ &\implies 3(x-1)(x-3) = 0 \\ &\implies x = 1 \text{ or } x = 3. \end{aligned}$$

Therefore the critical numbers for f are 1 and 3. Evaluating f at these numbers we get

$$f(1) = 6 \quad f(3) = 2$$

and evaluating f at the endpoints of the interval $[1, 5]$ gives

$$f(-1) = 4 \quad f(5) = 22.$$

Selecting the largest of these, we find the the absolute maximum of $f(x)$ is 22 and occurs at $x = 5$; the absolute minimum is 2 and occurs at $x = 3$.

Lec. #15

7.3 Applied Optimization

Example 7.8. A farmer has 400 m of fencing to build a rectangular enclosure with a partition parallel to one of the sides. What should the dimensions be so the maximum possible area is enclosed?

Example 7.9. A farmer want to build a rectangular enclosure with an area of $100\,\text{m}^2$. What should the dimensions be so the minimum length of fencing is used?

Example 7.10. A box with a square base and an open top must have a volume of $1000\,\text{cm}^3$. What should the dimensions be so that the minimum amount of material is used in the construction of the box.

Example 7.11. If $2400\,\text{cm}^2$ of cardboard is available to make a closed box with a square base, what should the dimensions be so the volume is as large as possible?

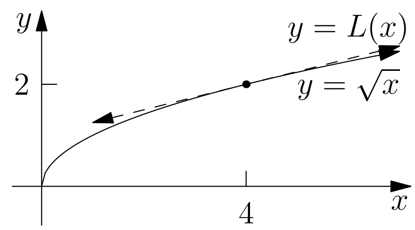
Example 7.12. Find the point on the graph of $y = x^2$ that is closest to the point $(1, 0)$.

8 Linear Approximation

Lec. #16

Suppose we want to estimate the value of $\sqrt{4.1}$ without a calculator. Since we know $\sqrt{4} = 2$, we suspect the answer should be slightly greater than 2. The problem is to estimate how much greater.

Note that this problem is equivalent to estimating the height of the graph of $y = \sqrt{x}$ at $x = 4.1$. Let the line tangent to the graph at $x = 4$ be given by $y = L(x)$. Because the graphs of $y = \sqrt{x}$ and $y = L(x)$ are very close (at least for values of x near 4, we should be able to estimate $\sqrt{4.1}$ by simply evaluating $L(4.1)$. For this reason, the function $L(x)$ is called the *linear approximation* of $f(x)$ based at $x = 4$.



To find the function $L(x) = mx + b$ we need the slope m which comes from evaluating the derivative at $x = 4$:

$$\frac{d}{dx}\sqrt{x} = \tfrac{1}{2}x^{-1/2} \implies m = \tfrac{1}{2}(4)^{-1/2} = \frac{1}{4}.$$

So we know $L(x) = \frac{1}{4}x + b$. To find b we force this line to go through the point $(4, 2)$:

$$2 = \tfrac{1}{4} \cdot 4 + b \implies b = 1.$$

So

$$L(x) = \tfrac{1}{4}x + 1$$

and we can use this to estimate

$$\sqrt{4.1} \approx L(4.1) = \tfrac{1}{4}(4.1) + 1 = 2.025$$

(compare this with the exact value $\sqrt{4.1} = 2.0248\cdots$)

In general, the linear approximation to a function $f(x)$, based at $x = x_0$, is

$$\boxed{L(x) = f(x_0) + f'(x_0)(x - x_0)}$$

Keep in mind: this is just the equation of the tangent line (in point-slope form) to the graph of $y = f(x)$ at x_0. The main idea behind the linear approximation is that $L(x)$ is close to $f(x)$, provided x is close to x_0.

Example 8.1. Use a linear approximation to estimate $\sqrt[3]{28}$.

What we want is a linear approximation of the function $f(x) = \sqrt[3]{x}$ based at $x = 27$ (since we know that $\sqrt[3]{27} = 3$). We have

$$f'(x) = \tfrac{1}{3}x^{-2/3} \implies f'(27) = \tfrac{1}{3}(27)^{-2/3} = \frac{1}{27}$$

so that

$$L(x) = f(27) + f'(27)(x - 27) = 3 + \tfrac{1}{27}(x - 27).$$

Now we can use this to get our approximation. Since 28 is close to 27 we have

$$\sqrt[3]{28} \approx L(28) = 3 + \tfrac{1}{27}(28 - 27) \approx 3.037$$

Example 8.2. Use a linear approximation to estimate (a) $\sqrt[4]{15.5}$ (b) $\ln(1.2)$ (c) $\sin(0.05)$

8.1 Differentials

Consider again the problem of estimating $\sqrt{4.1}$. An alternative approach, again based on the function $f(x) = \sqrt{x}$, is to write the derivative as

$$\frac{df}{dx} = \tfrac{1}{2}x^{-1/2}$$

and consider the expression $\frac{df}{dx}$ as the slope, i.e. "rise over run", between two points on the graph, a very small distance dx apart. That is, we can think of df as the infinitesimal rise corresponding to an infinitesimal "run" of dx. The "quantities" df and dx must be thought of as being infinitesimally small, and they are called *infinitesimals* or *differentials.*

Strictly speaking the equation above is only true in the limit $dx \to 0$. However, for finite but sufficiently small dx, it is still a good approximation:

$$\frac{df}{dx} \approx \tfrac{1}{2}x^{-1/2} \implies df \approx \tfrac{1}{2}x^{-1/2}\,dx.$$

Thus, when $x = 4$, if we increase x by an amount $dx = 0.1$ then we expect the value of $f(x)$ to increase by about

$$df = \tfrac{1}{2}(4)^{-1/2}\,(0.1) = 0.025$$

so we estimate that

$$\sqrt{4.1} \approx 2 + 0.025 = 2.025.$$

Example 8.3. Use differentials to estimate (a) $\sqrt[4]{15.5}$ (b) $\ln(1.2)$ (c) $\sin(0.05)$

Lec. #17

8.2 Error Analysis

Linear approximations and differentials are used often in error analysis of calculations based on experimental data.

Example 8.4. The width of a cube measured to be 13.0 cm with an uncertainty of 0.1 cm. Calculate the volume and the associated uncertainty.

The volume is clearly

$$V = x^3 = (13.0)^3 = 2197\,\text{cm}^3.$$

To estimate the uncertainty, consider: how much does the volume change if the true width is 13.1 cm (i.e. if our measurement of 13.0 cm is wrong by the full uncertainty amount)? We could calculate this exactly, but an approximation will be good enough (and simpler) so we'll use differentials:

$$\frac{dV}{dx} = 3x^2 \implies dV = 3x^2\,dx = 3(13.0)^2(0.1) \approx 51\,\text{cm}^3.$$

So we estimate that

$$V = 2197 \pm 51\,\text{cm}^3$$

Example 8.5. You measure the radius of a sphere to be 10 cm wtih an uncertainty of 2 cm. What is the volume, and what is the associated uncertainty.

Example 8.6. You measure height of a tree by standing 40 m from its base and measuring the angle of elevation of your line of sight to the top of the tree to be 30° with an uncertainty of 2°. What is the height of the tree and the associated uncertainty?

9 Newton's Method

Often when mathematics is applied to a problem in science, the solution of the problem is found by solving an equation. But what to do if you don't know how to solve the equation you need to solve? In fact, many equations simply *can't* be solved exactly. In situations like this, an approximate solution (accurate to, say, 6 decimal places) is all you really need. Newton's Method is one way of getting very accurate approximate solutions of equations.

Suppose we wanted to solve the equation

$$\cos x = x.$$

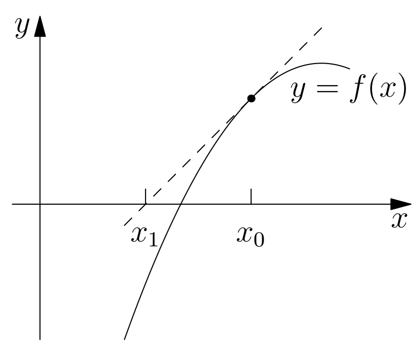
By rewriting this as

$$\underbrace{x - \cos x}_{f(x)} = 0$$

we can interpret the solution as the x-intercept the graph of $y = f(x)$.

From the graph of $f = f(x)$ we can read off a number close to the x-intercept; call this x_0. We can then find another point, x_1, that is even closer to the x-intercept by doing the following:

- construct the tangent line to the graph at x_0
- find the point where the tangent line crosses the x-axis; call this x_1.



If you go through the algebra, you will obtain the following formula for x_1 in terms of x_0:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

By repeating this process, constructing a tangent line at x_1 and locating its x-intercept at x_2, we obtain a point even closer to the x-intercept of the graph of $y = f(x)$:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In fact, we can repeat this process any number of times, with increasing accuracy, always iterating the formula

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad \text{for } n = 0, 1, 2, \ldots$$

Example 9.1. To solve the equation $f(x) = x - \cos x = 0$, we can start with an initial guess $x_0 = 1$ (which is close to the intersection of the graphs of $y = x$ and $y = \cos x$). Newton's method gives the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)}$$

so

$$x_1 = 1 - \frac{1-\cos(1)}{1+\sin(1)} \approx 0.75$$
$$x_2 = 0.75 - \frac{0.75-\cos(0.75)}{1+\sin(0.75)} \approx 0.7391$$
$$x_3 = 0.7391 - \frac{0.7391-\cos(0.7391)}{1+\sin(0.7391)} \approx 0.7391.$$

If we continue iterating Newton's method, the first 4 digits of x_n will never change, so the solution is $x = 0.7391$ accurate to 4 decimal places.

Example 9.2. Find solutions of each of the following equations, accurate to 3 decimal places:
(a) $x^2 - 2 = 0$ (b) $\sin x = 1 - x$

10 Shapes of Graphs

10.1 Increasing and Decreasing Functions

Lec. #18

Since the derivative gives the slope of $y = f(x)$, we can distinguish intervals where $f(x)$ is increasing from intervals where it is decreasing by examining the sign of $f'(x)$:

Theorem 10.1. *Let f be a differentiable function.*

- *If $f'(x) > 0$ on an interval then f is increasing on that interval.*
- *If $f'(x) < 0$ on an interval then f is decreasing on that interval.*

Example 10.1. Find the intervals on which $f(x)$ is increasing, and the intervals where it is decreasing:
(a) $f(x) = x^3 - 12x + 1$ (b) $f(x) = xe^{-x}$

At a critical point where f changes from increasing to decreasing, $f(x)$ must attain a maximum. Similarly, if f changes from decreasing to increasing, $f(x)$ has a minimum. This gives us the *first derivative test*:

Theorem 10.2 (First Derivative Test)**.** *Suppose c is a critical number for a continuous function f.*

- *If $f'(x)$ changes from positive to negative at c then f has a local maximum at c.*
- *If $f'(x)$ changes from negative to positive at c then f has a local minimum at c.*
- *If $f'(x)$ does not change sign at c then f has neither a maximum nor a minimum at c.*

10.2 Concavity

Given a function $f(x)$ and its derivative, $f'(x)$, we can differentiate $f'(x)$ to obtain the *second derivative* of $f(x)$.

Notation:

$$(f')' = f''(x) = \frac{d^2 f}{dx^2}$$

Differentiating again gives the *third derivative*:

$$(f'')' = f'''(x) = \frac{d^3 f}{dx^3}$$

In general, the n'th derivative is written

$$f^{(n)}(x) = \frac{d^n f}{dx^n}$$

The second derivative is closely related to the shape of the graph of $y = f(x)$. If $f''(x) > 0$ then we known $f'(x)$ is an *increasing* function. A graph whose slope is increasing is said to by *concave up*. On the other hand, if $f''(x) < 0$ then $f'(x)$ is decreasing, and $f(x)$ is *concave down*.

Example 10.2. Find the intervals of concavity for the following functions:
(a) $f(x) = x^3 - 12x + 1$ (b) $f(x) = xe^{-x}$

A number c such that $f''(c) = 0$ and $f''(x)$ changes sign at c is called an *inflection point* of f.

10.3 Curve Sketching

Armed with the intervals of increase, decrease and concavity for a given function, we have all the essentially information about the shape of its graph, and we can use this information to make an accurate sketch.

Example 10.3. Make tables of increase, decrease and concavity for the following funtions, and use this information to make a sketch of the graph of $y = f(x)$, labelling all local maxima and minima and inflection points.
(a) $f(x) = x^3 - 12x + 1$ (b) $f(x) = xe^{-x}$ (c) $f(x) = \dfrac{1}{1+x^2}$

11 L'Hôpital's Rule

Lec. #19

We have looked at many limit calculations similar to

$$\lim_{x\to 2} \frac{x^2-4}{x-2}.$$

This kind of limit is said to be *indeterminate form*, since we can't simply use the limit laws to arrive at the answer (because the denominator is approaching 0). In general, limits of indeterminate form are those that can be written

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where either

- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ (and we say the limit has the form "$\frac{0}{0}$"), or
- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$ (and we say the limit has the form "$\frac{\infty}{\infty}$").

The following simple rule can be useful for calculating such limits.

L'Hôpital's Rule: If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" and both f and g are differentiable at a, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example 11.1. $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$ has the form "$\frac{0}{0}$" so L'Hôpital's Rule can be used:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)} = \lim_{x\to 2} \frac{2x}{1} = 4.$$

Example 11.2. Use L'Hôpital's Rule to evaluate:

(a) $\lim_{x\to\infty} \frac{x}{e^x}$ (b) $\lim_{x\to\infty} x^2 e^{-x}$ (c) $\lim_{x\to\infty} x^{1000} e^{-x}$ (d) $\lim_{x\to 1} \frac{\ln x}{x-1}$ (e) $\lim_{x\to 0} \frac{x^2}{\ln x}$

L'Hôpital's Rule is often used to find horizontal asymptotes of graphs:

Definition 11.1. The line $y = b$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

For example, we found above the $\lim_{x\to\infty} xe^{-x} = 0$ so the graph of $y = xe^{-x}$ has the line $y = 0$ as a horizontal asymptote.

Example 11.3. Find intervals of increase, decrease and convavity, as well as all horizontal and vertical asymptotes for $y = xe^{-x}$, and use this to sketch the graph.

12 Parametric Curves

Lec. #20

We have been dealing with curves defined by the graph of some function $y = f(x)$. But often in the sciences curves are described not by the graphs of functions, by *parametrically.* The following is a familiar example from physics.

Example 12.1. A ball is thrown horizontally from the top of the building, at speed 5 m/s. At time t its x- and y-coordinates (in meters), relative to the top of the building, are

$$x = 5t, \qquad y = -4.9t^2.$$

The two equations above define the path of the ball as a *parametric curve.* That is, for each value of t we can calculate and plot the position of the ball; by doing this for all values of $t \geq 0$ we obtain the curve representing the trajectory of the ball.

In general, a parametric curve in two dimensions is described by equations

$$x = f(t)$$
$$y = g(t)$$

giving the x- and y-coordinates of a point on the curve at "time" t. The variable t is a parameter, and it label is arbitrary (it might as well be s or something else; this doesn't change the curve). This description of a curve arises often in physics.

Problem: How to find the slope of a parametric curve?

Answer: For a curve defined by parametric equations $x = f(t)$, $y = g(t)$, the slope at the point $\big(x(t), y(t)\big)$ is given by

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \quad (\text{if } dx/dt \neq 0.)}$$

Example 12.2. For the previous example, find the slope of the ball's trajectory at time $t = 2$.

Example 12.3. The path of a ball thrown upward at an angle to the ground is given by the parametric equations

$$x(t) = 5t$$
$$y(t) = 2 - 4.9t^2.$$

(a) Find an expression for dy/dx.
(b) At what time does the ball reach its maximum height?

Example 12.4. For the parametric curve

$$x(t) = 2\sin 2t$$
$$y(t) = 2\sin t$$

(a) find the slope at the point where $t = 5$.
(b) find all points where the curve has a horizontal tangent.