THOMPSON RIVERS UNIVERSITY

MATH 114
Calculus I

Instructor: Richard Taylor

MIDTERM EXAM #2
SOLUTIONS

27 March 2007 11:30–13:20

PROBLEM	GRADE	OUT OF
1		12
2		5
3		4
4		8
5		4
6		6
7		9
8		4
TOTAL:		52

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 7 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

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Problem 1: Find the derivative of each of the following functions. *Do not simplify your answers.*

(a) $f(x) = (\sin x)(\ln x)$

SOLUTION:

$$f'(x) = (\cos x)(\ln x) + (\sin x)\frac{1}{x}$$

(b) $g(x) = \arccos(x^2)$

SOLUTION:

$$g'(x) = \frac{-2x}{\sqrt{1-x^4}}$$

(c) $h(x) = \ln(\arctan x)$

SOLUTION:

$$h'(x) = \frac{1}{\arctan x} \cdot \frac{1}{1+x^2}$$

(d) $f(x) = x^{1/x}$

SOLUTION:

logarithmic differentiation: $y = x^{1/x} \implies \ln y = \ln(x^{1/x}) = \frac{1}{x}\ln x$

$$\implies \frac{1}{y} \cdot y' = -\frac{1}{x^2}\ln x + \frac{1}{x} \cdot \frac{1}{x}$$

$$\implies \boxed{f'(x) = x^{1/x}\left(-\frac{\ln x}{x^2} + \frac{1}{x^2}\right)}$$

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Problem 2: Use implicit differentiation to find the slope of the graph of

$$x^2y + y^2x = 6$$

at the point $(1, 2)$.

SOLUTION:

$$2xy + x^2y' + 2yy'x + y^2 = 0$$

$$\implies 2(1)(2) + (2)^2y' + 2(2)y'(1) + (2)^2 = 0 \implies 8 + 5y' = 0$$

$$\implies \boxed{y' = -\frac{8}{5}}$$

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Problem 3: The diameter of a sphere is measured as $100 \pm 1\,\mathrm{cm}$ and the volume is calculated from this measurement. What is the sphere's volume and the associated uncertainty?

SOLUTION:

$$V = \tfrac{4}{3}\pi r^3 = \tfrac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{\pi}{6}D^3$$

$$\implies dV = \frac{\pi}{6}3D^2\,dD = \frac{\pi}{2}D^2\,dD$$

$$\text{so}\quad V = \frac{\pi}{6}(100)^3 \approx 523599\,\mathrm{cm}^3$$

$$\text{and}\quad dV \approx \frac{\pi}{2}(100)^2(1) \approx 15708\,\mathrm{cm}^3$$

$$\implies \boxed{V = 524 \pm 16 \times 10^3\,\mathrm{cm}^3}$$

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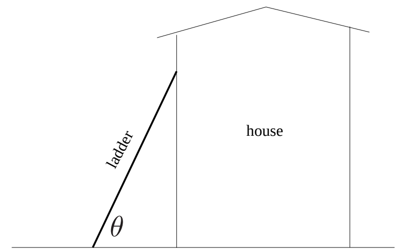
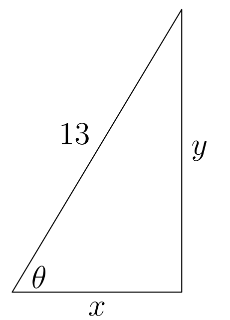
Problem 4: A 13-foot ladder is leaning against a house when its base starts to slide away, so that the top of the ladder slides down the wall. The top of the ladder remains in contact with the wall at all times. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec. At this instant:

(a) at what rate is the top of the ladder sliding down the wall?



SOLUTION:



$x^2 + y^2 = 13^2 \implies 2xx' + 2yy' = 0$

when $x = 12$, $y = \sqrt{13^2 - 12^2} = 5$

$\implies 2(12)(5) + 2(5)y' = 0 \implies \boxed{y' = -12\,\text{ft/s}}$

(b) at what rate is the angle θ between the ladder and the ground changing?

SOLUTION:

$$\cos\theta = \frac{x}{13} \implies (-\sin\theta)\theta' = \frac{x'}{13}$$

$$\implies \left(-\frac{5}{13}\right)\cdot\theta' = \frac{5}{13} \implies \boxed{\theta' = -1\,\text{rad/s}}$$

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Problem 5: Use a linear approximation to estimate $\sqrt[3]{1.009}$.

SOLUTION:

linearize $f(x) = x^{1/3}$ at $x = 1$:

$$f'(x) = \tfrac{1}{3}x^{-2/3} \implies f'(1) = \tfrac{1}{3}(1)^{-2/3} = \tfrac{1}{3}$$

$$\implies L(x) = f(1) + f'(1)\cdot(x-1) = 1 + \tfrac{1}{3}(x-1)$$

$$\text{so} \quad \sqrt[3]{1.009} = f(1.009) \approx L(1.009) = 1 + \tfrac{1}{3}(1.009 - 1) = \boxed{1.003}$$

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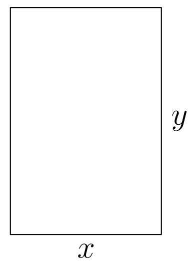
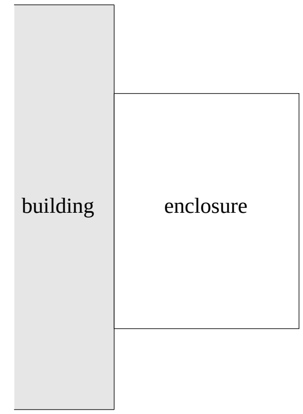
Problem 6: A rectangular enclosure is to be formed adjacent to a building, so that the wall of the building forms one side of the enclosure. The other three sides are to be fenced. (See the diagram below.) With 800 m of fencing available, what should be the dimensions of the enclosure so that it has the largest possible area?





SOLUTION:

$$2x + y = 800 \implies y = 800 - 2x$$

Want to maximize:

$$A = xy = x(800 - 2x) = 800x - 2x^2 = A(x) \quad \text{on the interval } [0, 400]$$

Critical points:

$$A' = 800 - 4x = 0 \implies x = 200$$

Compare values:

$$A(0) = 0$$
$$A(200) = 80000\,\text{m}^2$$
$$A(400) = 0$$

The dimensions giving the largest possible area are $\boxed{x = 200\,\text{m},\ y = 400\,\text{m}.}$

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Problem 7: Consider the function $f(x) = -2x^3 + 6x^2 - 3$.

(a) Find the intervals of increase and decrease for $f(x)$.

SOLUTION:

$$f' = -6x^2 + 12x = -6x(x-2)$$
$$f'(x) = 0 \implies x = 0, 2 \quad \text{are the critical points for } f$$

interval	f'	f
$(-\infty, 0)$	$-$	decreasing
$(0, 2)$	$+$	increasing
$(2, \infty)$	$-$	decreasing

$$\Longrightarrow \begin{cases} \text{local min at } (0, -3) \\ \text{local max at } (2, 5) \end{cases}$$

(b) Find the intervals of concavity for $f(x)$.

SOLUTION:

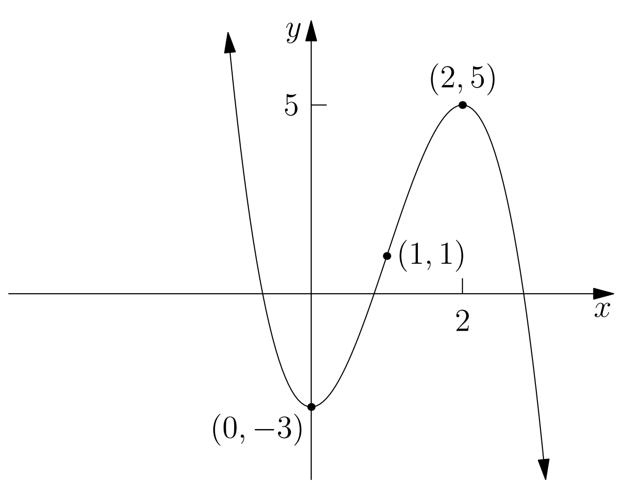
$$f'' = -12x + 12 = -12(x-1)$$
$$f''(x) = 0 \implies x = 1 \quad \text{is the only critical point for } f'$$

interval	f''	f
$(-\infty, 1)$	$+$	concave up
$(1, \infty)$	$-$	concave down

$\Longrightarrow$ inflection point at $(1, 1)$

(c) Sketch a graph of $y = f(x)$ on the axes below. Label the coordinates of all local extrema and inflection points.

SOLUTION:



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Problem 8: Use Newton's method to solve $x^4 = 2$ accurate to 2 decimal places.

SOLUTION:

$$\underbrace{x^4 - 2}_{f(x)} = 0$$

Newton's method:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^4 - 2}{4x_n^3} \end{aligned}$$

Initial approximation: $x_1 = 1.2$ (since $1.2^4 \approx 2.07 \approx 2$)

$$\begin{aligned} &\Longrightarrow x_2 = 1.2 - \frac{(1.2)^4 - 2}{4(1.2)^3} \approx 1.1894 \\ &\Longrightarrow x_3 = 1.1894 - \frac{(1.1894)^4 - 2}{4(1.1894)^3} \approx 1.1892 \end{aligned}$$

So to 2 decimal places (rounded) $\boxed{x = 1.19}$