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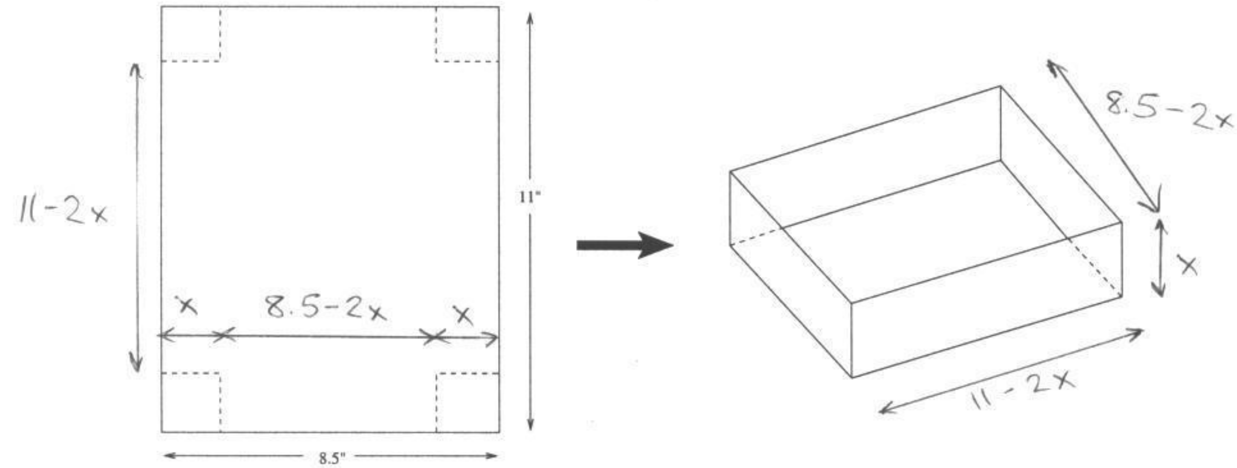
Problem 1: Squares are cut out of the corners of a standard 8.5" × 11" sheet of paper. The sides are then folded up to form a container (a box with no top) as shown in the diagram below. What is the maximum possible volume of such a container?



maximize: $V(x) = (11-2x)(8.5-2x)x$ for $x \in [0, 4.25]$

$= 93.5x - 39x^2 + 4x^3$

critical pts:

$V'(x) = 93.5 - 78x + 12x^2 = 0$

$\rightarrow x = \dfrac{78 \pm \sqrt{78^2 - 4(12)(93.5)}}{2(12)} = 4.91 \text{ or } 1.59$

outside $[0, 4.25]$ so reject

compare:

$V(0) = 0$

$V(4.25) = 0$

$V(1.59) = 7.83'' \times 5.33'' \times 1.59'' = \boxed{66.1 \text{ in}^3}$

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Problem 2: A rock is dropped into a lake and an expanding circular ripple results. When the radius of the ripple is 8 m, the radius is increasing at a rate of 0.5 m/s. At what rate is the area enclosed by the ripple changing at this time?

$$A = \pi r^2$$

$$\rightarrow A' = 2\pi r r'$$

$$= 2\pi(8)(0.5)$$

$$= \boxed{8\pi \approx 25.1 \text{ m}^2/\text{s}}$$

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Problem 3: The period of a pendulum is given by the formula $T = 2\pi\sqrt{L/g}$ where L is the length of the pendulum in meters, $g = 9.81\,\text{m/s}^2$ is the acceleration due to gravity, and T is the pendulum's period in seconds. If the length of the pendulum is measured with an accuracy of 5%, what is the uncertainty (in %) in the calculated period?

$$T = \frac{2\pi}{\sqrt{g}} L^{1/2} \rightarrow dT = \frac{2\pi}{\sqrt{g}} \frac{1}{2} L^{-1/2} dL$$

$$\text{so } \frac{dT}{T} = \frac{\frac{2\pi}{\sqrt{g}} \frac{1}{2} L^{-1/2} dL}{\frac{2\pi}{\sqrt{g}} L^{1/2}} = \frac{1}{2} \underbrace{\frac{dL}{L}}_{0.05}$$

$$\therefore \frac{dT}{T} \approx \frac{1}{2}(0.05) = 0.025 = \boxed{2.5\%}$$

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Problem 4: Consider the function $f(x) = x^2 e^{-x}$.

(a) Determine the intervals of increase and decrease for this function.

$$f'(x) = 2xe^{-x} - x^2e^{-x}$$
$$= x(2-x)e^{-x}$$

$$f' = 0 \Rightarrow x = 0 \text{ or } 2$$

interval	f'	f
$(-\infty, 0)$	$-$	decr.
$(0, 2)$	$+$	incr.
$(2, \infty)$	$-$	decr.

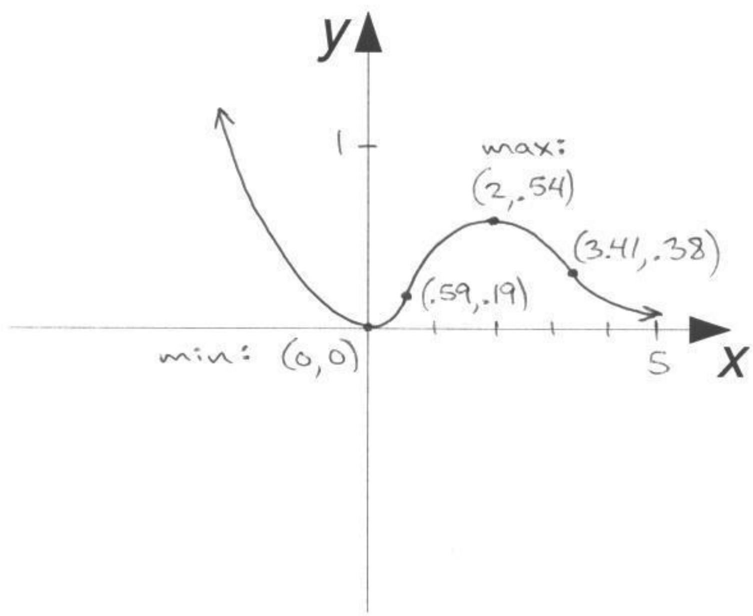
(b) Determine the intervals of concavity.

$$f''(x) = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}$$
$$= (x^2 - 4x + 2)e^{-x}$$

$$f' = 0 \Rightarrow x = \frac{4 \pm \sqrt{4^2 - 8}}{2}$$
$$= 2 \pm \sqrt{2}$$

interval	f''	f
$(-\infty, 2-\sqrt{2})$	$+$	concave up
$(2-\sqrt{2}, 2+\sqrt{2})$	$-$	concave down
$(2+\sqrt{2}, \infty)$	$+$	concave up

(c) Sketch a graph of $y = f(x)$. Label all maximum and minimum values and inflection points.



max. at $(2, 0.54)$
min. at $(0, 0)$

infl. pts. at
$(0.59, 0.19)$
$(3.41, 0.38)$

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Problem 5: Consider the graph defined by the equation

$$x \tan^{-1} y = x^2 + y.$$

Find the slope of the tangent line to the graph at the point $(0, 0)$.

implicit differentiation:

$$(1) \tan^{-1} y + x \frac{1}{1+y^2} \cdot y' = 2x + y'$$

substitute $x=0$, $y=0$:

$$\underbrace{\tan^{-1}(0)}_{0} + 0 \cdot \frac{1}{1+0^2} y' = 0 + y'$$

$$\rightarrow \boxed{y' = 0}$$

/4 **Problem 6:** Use a linear approximation or differentials to estimate $\sqrt[4]{83}$. (Note that $\sqrt[4]{81} = 3$.)

- let $y(x) = \sqrt[4]{x} = x^{1/4}$

  so $dy = \frac{1}{4} x^{-3/4} dx$.

- at $x = 81$, $dx = 2$ so:

  $y(81) = 81^{1/4} = 3$

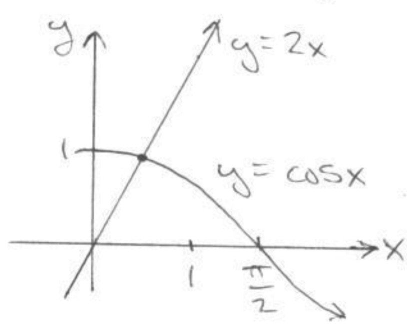
  $dy = \frac{1}{4}(81)^{-3/4}(2) = \frac{1}{4} \cdot \frac{1}{27} \cdot 2 = \frac{1}{54} \approx 0.0185$

$\therefore \sqrt[4]{83} \approx 3 + 0.0185 = \boxed{3.0185}$

/4 **Problem 7:** Use Newton's method to find a solution of $\cos x = 2x$ accurate to 3 decimal places.

- $\underbrace{\cos x - 2x}_{f(x)} = 0$

  $f'(x) = -\sin x - 2$



- Newton's method:

  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + \frac{\cos x_n - 2x_n}{\sin x_n + 2}$

- guess $x_0 = 0.6$ (from graph)

  $\rightarrow x_1 = 0.6 + \frac{\cos(0.6) - 2(0.6)}{\sin(0.6) + 2} = 0.4539$

  $\rightarrow x_2 = 0.4502$

  $\rightarrow x_3 = 0.4502$

$\therefore \boxed{x \approx 0.450}$