Name:______________________________

Signature:______________________________

Instructions:

1. Prepare your solutions on your own paper, with this exam paper stapled to the front.
2. It is permissible to discuss the problems with other students or faculty members. However, your written solutions must be exclusively your own work. The TRU Policy on Academic Integrity (`http://www.tru.ca/__shared/assets/ed05-05657.pdf`) will be strictly enforced.
3. You may use any tools at your disposal (e.g. calculator, software, Wolfram Alpha) to simplify calculations, but be careful to indicate where and why you have done so. In any case, you must clearly explain your reasoning to receive full credit.
4. Organization and neatness count. A lot.
5. Submit your solutions to my office or to Marcy Desrosiers in the Mathematics & Statistics Department reception area **before 12:00PM, 29 April 2017**.
6. Late submissions will not be accepted.

/12

Problem 1: Consider the integral function

$$J[y] = \int_1^2 \frac{y'(x)^2}{x}\,dx.$$

The integrand here is strongly convex (on an appropriately defined set). Find the unique $y \in D$ that minimizes $J[y]$ over D, for the following cases. In each case, is the minimizer (if it exists) unique?

(a) $D = \{y \in C^1[1,2] : y(1) = 0,\ y(2) = 3\}$

(b) $D = \{y \in C^1[1,2] : y(2) = 3\}$

(c) $D = C^1[1,2]$

/8

Problem 2: Let $\delta(x)$ be the Dirac delta function. Justify the identity

$$\delta(1-x^2) = \frac{\delta(x-1)+\delta(x+1)}{2}.$$

Hint: consider the integral $\displaystyle\int_{-\infty}^{\infty} \delta(1-x^2)\,dx$.

/12

Problem 3: Let $f : [0, 1] \to \mathbb{R}$ be a given continuous function, and consider the following differential equation for the function y:

$$\begin{cases} y'''(x) = f(x) & 0 \le x \le 1 \\ y(0) = y'(0) = y''(0) = 0. \end{cases} \tag{1}$$

Intuitively, finding $y(x)$ requires three integrations. This problem illustrates that by employing a Green's function we can actually find y via a *single* integral.

(a) Show that if the Green's function $u = G(x; s)$ satisfies

$$\begin{cases} u''' = \delta(x - s) & 0 \le x \le 1 \\ u(0) = u'(0) = u''(0) = 0. \end{cases}$$

then the function

$$y(x) = \int_0^1 f(s) G(x; s)\, ds \tag{2}$$

satisfies (1).

(b) Find the Green's function $G(x; s)$ for this problem.

(c) For the case $f(x) = x$, solve (1) directly by integrating thrice. Verify that (2) gives the same result.

/12

Problem 4: The Fourier transform is often useful for finding Green's functions. Consider the following partial differential equation for the function $u(x, t)$:

$$\begin{cases} u_{tt} - u_{xx} + u = 0 & -\infty < x < \infty \\ u(x, 0) = \delta(x - s) & s \in \mathbb{R} \\ u_t(x, 0) = 0. \end{cases}$$

(a) Apply a Fourier transform with respect to x to obtain a 2nd-order ordinary differential equation for the function $\hat{u}(\omega, t)$.

(b) Find $\hat{u}(\omega, t)$ by solving the differential equation from part (a).

(c) Invert the Fourier transform to obtain $u(x, t)$. If you cannot invert the transform analytically, you can at least express u as an integral.

(d) **Bonus:** Use the method of stationary phase to approximate the integral in (c), thereby giving an approximation of $u(x, t)$ valid for $x \gg 1$.

/8

Problem 5: Consider the cubic equation

$$x^3 - x + \varepsilon = 0 \tag{3}$$

with $|\varepsilon| \ll 1$. Assume a series solution of the form

$$x(\varepsilon) = x_0 + x_1 \varepsilon + x_2 \varepsilon^2 + \cdots .$$

Substitute this into (3) and consider the limit $\varepsilon \to 0$ to determine the coefficients x_0, x_1, x_2 and thereby obtain a three-term asymptotic approximation for each of the three roots. Verify that your series gives reasonably accurate results for the cases $\varepsilon = 0.1$ and $\varepsilon = 0.01$.